Fractional Perfect b-Matching Polytopes

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Full details: RB *Fractional perfect b-matching polytopes*
In preparation

Outline

(1) Define fractional perfect b-matching polytopes

(2) Obtain theorems for the dimensions, faces and vertices of such polytopes

(3) Present well-studied examples of such polytopes

Fractional Perfect b-Matching Polytope

Let:

- $G =$ undirected graph with vertex set V, edge set E
  (loops & multiple edges allowed)
- $b =$ assignment of a positive number to each vertex of G

Define the *fractional perfect b-matching polytope of G* as:

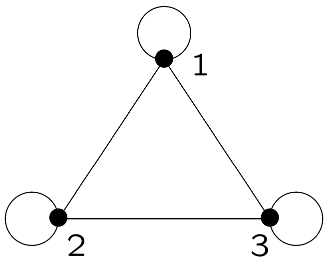
- $\mathcal{P}(G,b) = \left\{ x \in \mathbb{R}^E \,\middle|\, x_e \geq 0 \text{ for all } e \in E, \sum_{e \in \delta(v)} x_e = b_v \text{ for all } v \in V \right\}$

  $=$ set of all assignments of nonnegative numbers to the edges of G such that the sum of numbers over all edges incident to a vertex v is the prescribed number b_v for that vertex

  [a loop is considered as incident once with its associated vertex]

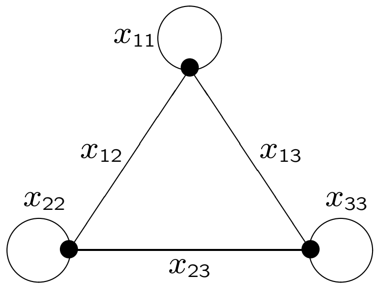
Example

$G =$



$b = (b_1, b_2, b_3) = (1, 1, 1)$

$\Rightarrow \mathcal{P}(G, b) = \Bigg\{$



$$\Bigg| \begin{array}{l} x_{11},\, x_{12},\, x_{13},\, x_{22},\, x_{23},\, x_{33} \geq 0, \\ x_{11} + x_{12} + x_{13} = 1, \\ x_{12} + x_{22} + x_{23} = 1, \\ x_{13} + x_{23} + x_{33} = 1 \end{array} \Bigg\}$$

(Corresponds to polytope of 3×3 symmetric doubly stochastic matrices)

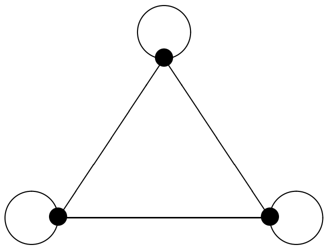
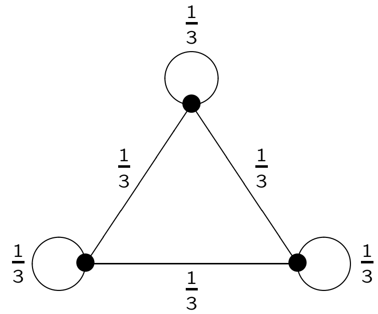
Dimension of $\mathcal{P}(G,b)$

Theorem. *If $\mathcal{P}(G,b)$ contains a completely positive point (i.e., if there is $x \in \mathcal{P}(G,b)$ with $x_e > 0$ for each $e \in E$), then $\mathcal{P}(G,b)$ has dimension $|E| - |V| + B$, where B is the number of bipartite components of G.*

Proof

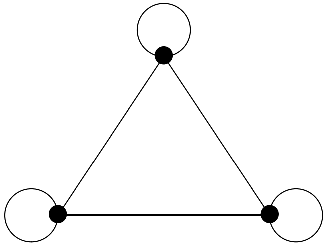
- $\mathcal{P}(G,b) = \{x \in \mathbb{R}^E \mid I_G\, x = b, \quad x_e \geq 0 \text{ for each } e \in E\}$, where I_G is the incidence matrix of G
- Therefore, if $\mathcal{P}(G,b)$ contains a completely positive point, then
$$\begin{aligned}\dim \mathcal{P}(G,b) &= \text{dim(kernel of } I_G) \\ &= (\#\text{ columns of } I_G) - (\#\text{ rows of } I_G) + \text{dim(kernel of } {I_G}^T) \\ &= |E| - |V| + \text{dim(kernel of } {I_G}^T)\end{aligned}$$
- dim(kernel of ${I_G}^T$) = (# bipartite components of G), since
$$\begin{aligned}{I_G}^T y = 0 \quad &\iff \quad y_{v_1} + y_{v_2} = 0 \text{ for all adjacent vertices } v_1 \;\&\; v_2 \\ &\iff \quad y_v = \begin{cases} 0, & v \text{ a vertex of a nonbipartite component} \\ \lambda_C, & v \text{ in colour class 1 of bipartite component } C \\ -\lambda_C, & v \text{ in colour class 2 of bipartite component } C \end{cases}\end{aligned}$$
with λ_C arbitrary for each C

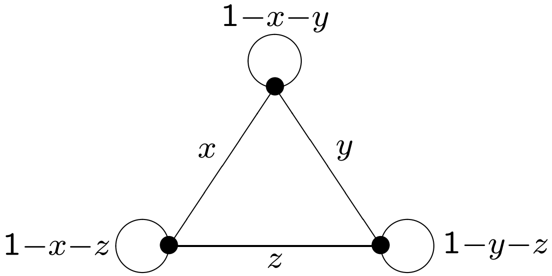
Example

$\mathcal{P}\Bigg($  $, (1,1,1)\Bigg)$ has completely positive point 

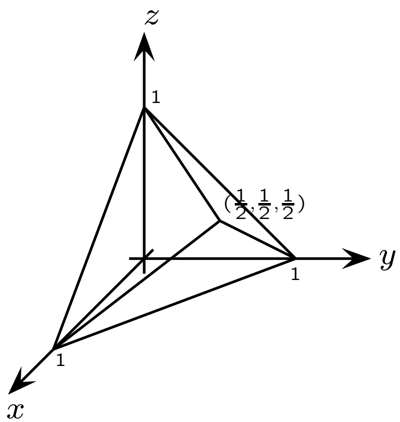
$$\begin{aligned}\text{Therefore, dimension} &= (\#\text{ edges}) - (\#\text{ vertices}) + (\#\text{ bipartite components})\\ &= 6 - 3 + 0\\ &= 3\end{aligned}$$

Explicitly:

$\mathcal{P}\Bigg($  $,\ (1,1,1)\Bigg)$

$\cong \Bigg\{$ 

$\Bigg|\ x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y \leq 1,\ x+z \leq 1,\ y+z \leq 1 \Bigg\}$

$\cong$ 

Faces of $\mathcal{P}(G, b)$

- Let: F = a nonempty face of $\mathcal{P}(G, b)$
- Define: the graph G_F of F as the graph obtained from G by deleting each edge e for which $x_e = 0$ for all $x \in F$

Then:

- F is affinely isomorphic to $\mathcal{P}(G_F, b)$
- $\dim F$ = (# edges of G_F) − $|V|$ + (# bipartite components of G_F) (since $\mathcal{P}(G_F, b)$ necessarily contains a completely positive point)

Vertices of $\mathcal{P}(G,b)$

- Let: $x =$ a point of $\mathcal{P}(G,b)$
- Define: the graph G_x of x as the graph obtained from G by deleting each edge e for which $x_e = 0$

Theorem. *A point $x \in \mathcal{P}(G,b)$ is a vertex of $\mathcal{P}(G,b)$ if and only if each component of G_x either is acyclic or else contains a unique cycle with that cycle having odd length.*

[Note: loop = cycle of length 1; 2 vertices connected by 2 edges = cycle of length 2]

<u>Proof</u>

x is a vertex of $\mathcal{P}(G,b)$

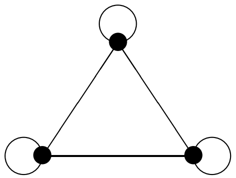
$\iff$ $\dim \mathcal{P}(G_x,b) = 0$

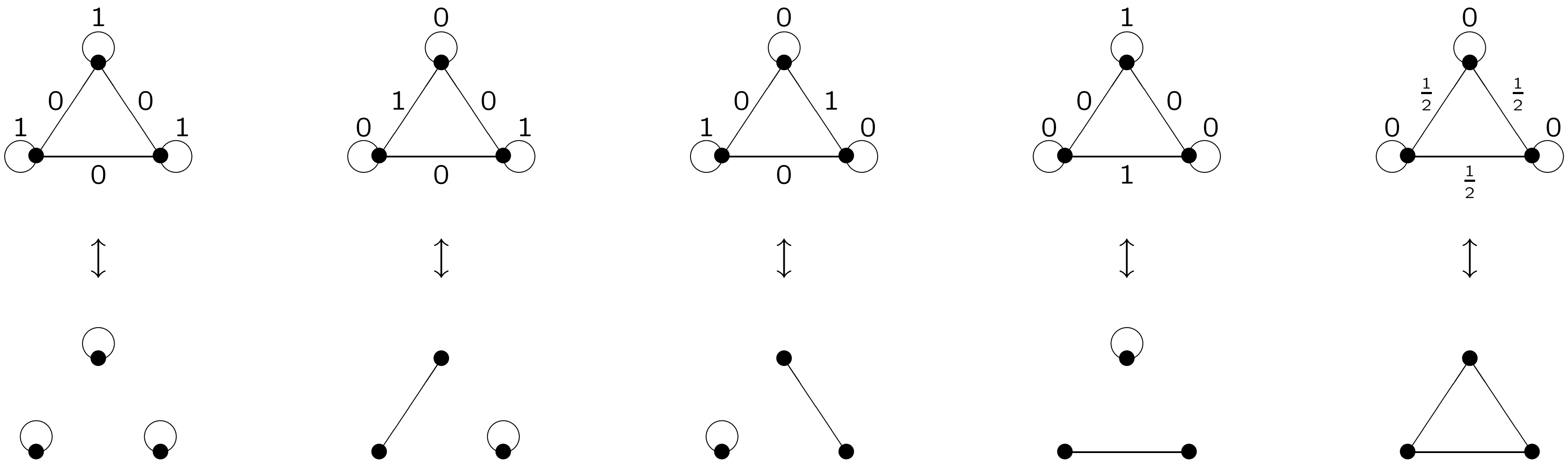
$\iff$ each component C of G_x satisfies
  (# edges in C) $-$ (# vertices in C) $+ 1 = 0$ if C is bipartite
  or (# edges in C) $-$ (# vertices in C) $= 0$ if C is nonbipartite

$\iff$ each component of G_x is acyclic
  or has a unique cycle, the length of which is odd
  [using standard results for bipartite graphs, trees & connected graphs with unique cycles]

Corollary. *For G bipartite, $x \in \mathcal{P}(G,b)$ is a vertex of $\mathcal{P}(G,b)$ if and only if G_x is a forest.*

Example

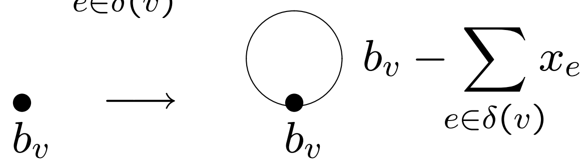
Vertices of $\mathcal{P}\Bigg($  $, (1,1,1)\Bigg)$ are:



Standard Polytopes of Type $\mathcal{P}(G,b)$

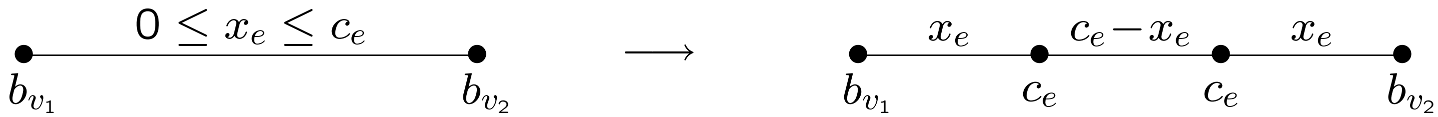
- *Polytopes in which* $\sum_{e\in\delta(v)} x_e = b_v$ *is replaced by* $\sum_{e\in\delta(v)} x_e \le b_v$ *for certain* $v \in V$.

  Attach extra loop to each such vertex v:



- *Polytopes which include conditions* $x_e \le c_e$ *for certain* $e \in E$, *where* c_e *is a prescribed positive number for edge* e.

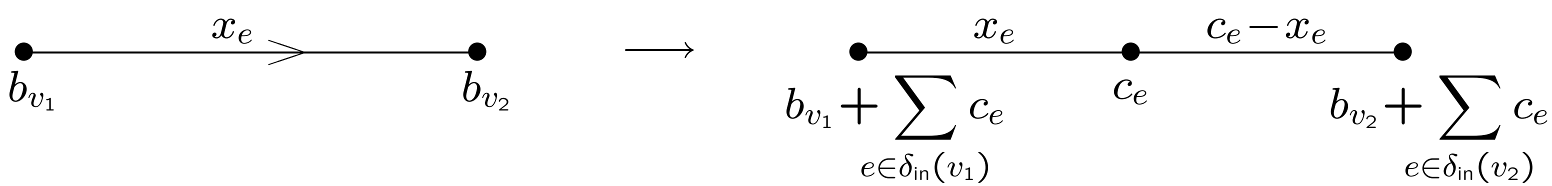
  Insert 2 extra vertices into each such edge (Tutte 1954):



- *Polytopes of flows or transshipments, i.e.,* G *is a directed graph,*

  $0 \le x_e \le c_e$ *for all* $e \in E$, $\sum_{e\in\delta_{\text{out}}(v)} x_e - \sum_{e\in\delta_{\text{in}}(v)} x_e = b_v$ *for all* $v \in V$.

  Insert extra vertex into each edge (Orden 1956; Ford, Fulkerson 1962):



- *Faces of $\mathcal{P}(G,b)$.*
  Already considered.

- *Certain other matching-type polytopes.* (see e.g. Schrijver 2003)
  e.g. $\mathcal{P}(G,b)$ with $b_v = 1$ for all v is the fractional perfect matching polytope of G; for G bipartite, it is the perfect matching polytope of G.

- *Polytopes of magic labelings of graphs* (e.g. Stanley 1973, 1976; Ahmed 2008).
  $\mathcal{P}(G,b)$ with $b_v = 1$ for all v is the polytope of magic labelings of G.

- *Symmetric transportation polytope $\mathcal{N}(R)$ and related polytopes $\mathcal{N}(\leq R)$, $\mathcal{N}_{\leq W}(R)$ & $\mathcal{N}_{\leq W}(\leq R)$* (e.g. Brualdi 1976, 2006).

$$\begin{aligned}\mathcal{N}(R) &= \{\text{symmetric nonnegative } n\times n \text{ matrices with line sum vector } R\}\\ &\cong \mathcal{P}(G,R), \text{ where } G \text{ is the complete graph } K_n \text{ with a loop at each vertex}\end{aligned}$$

- *Transportation polytope $\mathcal{N}(R,S)$ and related polytopes $\mathcal{N}(\leq R, \leq S)$, $\mathcal{N}_{\leq W}(R,S)$ & $\mathcal{N}_{\leq W}(\leq R, \leq S)$* (e.g. Brualdi 1976, 2006).

$$\begin{aligned}\mathcal{N}(R,S) &= \{\text{nonnegative } m\times n \text{ matrices with row \& column sum vectors } R \,\&\, S\}\\ &\cong \mathcal{P}(G,(R,S)), \text{ where } G \text{ is the complete bipartite graph } K_{m,n}\end{aligned}$$

- *Polytope of doubly stochastic matrices (Birkhoff or assignment polytope)* (e.g. Birkhoff 1946).
  $\mathcal{B}_n = \{$nonnegative $n \times n$ matrices with all line sums 1$\}$
  $\cong \mathcal{P}(G, b)$, where $G = K_{n,n}$ & $b_v = 1$ for all v

- *Certain polytopes related to polytope of doubly stochastic matrices.* (see e.g. Brualdi 2006)
  e.g. polytope of doubly substochastic matrices
  polytope of extensions of doubly substochastic matrices
  polytope of symmetric doubly stochastic matrices
  polytope of symmetric doubly substochastic matrices
  polytope of tridiagonal doubly stochastic matrices

- *Alternating sign matrix polytope* (RB, Knight 2007; Striker 2009).
  $\mathcal{A}_n = \{$all $n \times n$ real matrices with all complete line sums 1 &
  all partial line sums extending from each end of the line nonnegative$\}$
  $\cong \mathcal{P}(G, b)$, where G = certain $n \times n$ grid-type graph & $b_v \in \{1, 2\}$ for all v

- *Polytope of all points of $\mathcal{P}(G, b)$ invariant under the action of a group H of automorphisms of G* (RB—in preparation).
  $G \to G/H = (V/H, E/H) =$ 'orbit graph of G'