Osculating Paths and Oscillating Tableaux

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Full details: RB *Osculating paths and oscillating tableaux*
Electronic J. Combinatorics 15 (2008) #R7 (60 pages)

Main Result

- There is a bijection between certain tuples of *osculating paths* and certain *generalized oscillating tableaux*

Motivation

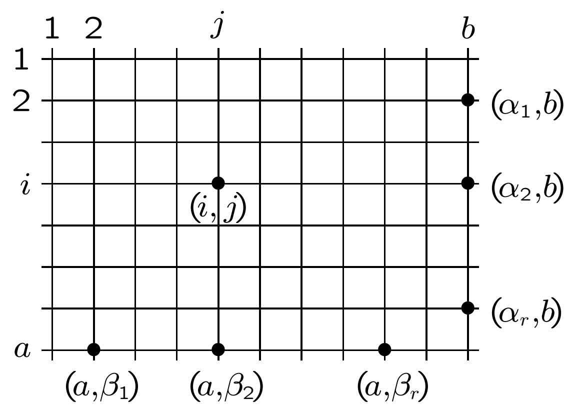
- Generalize well-known bijections between certain tuples of *nonintersecting paths* and *semistandard Young tableaux*
- Improve understanding of combinatorics of *alternating sign matrices*

Osculating Paths

The underlying configuration:

- a by b rectangle of points with
  - rows labeled 1 to a from top to bottom
  - columns labeled 1 to b from left to right
  - the point in row i and column j labeled (i,j)

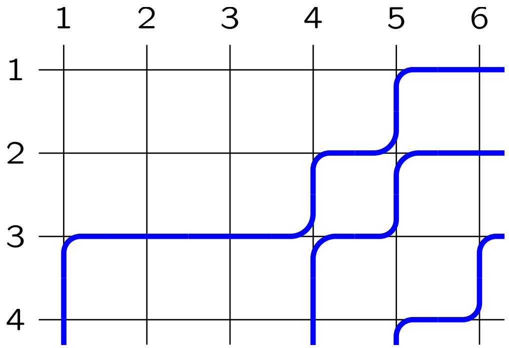
- r points chosen on lower boundary, $(a,\beta_1),\ \ldots,\ (a,\beta_r)$,
  r points chosen on right boundary, $(\alpha_1,b),\ \ldots,\ (\alpha_r,b)$,
  for some $\{\beta_1,\ldots,\beta_r\}=\beta$, $\{\alpha_1,\ldots,\alpha_r\}=\alpha$ with $\beta_1<\ldots<\beta_r$, $\alpha_1<\ldots<\alpha_r$



Let $\mathrm{OP}(a, b, \alpha, \beta)$ be the set of all r-tuples of paths in which

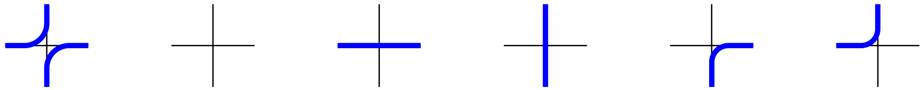
- the k-th path of a tuple starts at (a, β_k) and ends at (α_k, b)
- each path of a tuple can take only unit steps up or right
- different paths of a tuple are allowed to meet at lattice points, i.e. *osculate*, but not cross or share lattice edges

e.g.



$\in \mathrm{OP}(4, 6, \{1, 2, 3\}, \{1, 4, 5\})$

$\Rightarrow$ Six possible vertex configurations:



Alternating Sign Matrices

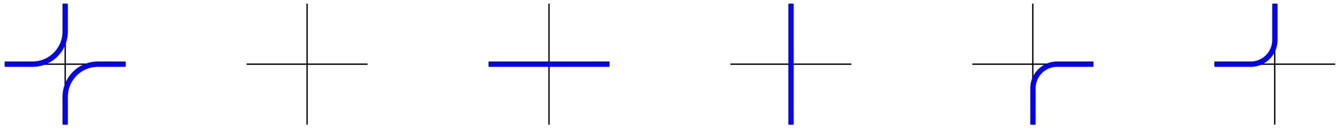
Let $\mathrm{ASM}(a, b, \alpha, \beta)$ be the set of all $a \times b$ matrices A for which

- each entry of A is 0, -1 or 1
- along each row and column of A the nonzero entries, if there are any, alternate in sign starting with a 1
- $\sum_{j=1}^{b} A_{ij} = \delta_{i\in\alpha}\,, \quad i = 1, \ldots, a$
- $\sum_{i=1}^{a} A_{ij} = \delta_{j\in\beta}\,, \quad j = 1, \ldots, b$

e.g. $$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \in \mathrm{ASM}(4, 6, \{1, 2, 3\}, \{1, 4, 5\})$$

$\Rightarrow \mathrm{ASM}(n, n, [n], [n]) = \{\text{standard } n \times n \text{ ASMs}\}$ (with $[n] \equiv \{1, \ldots, n\}$)

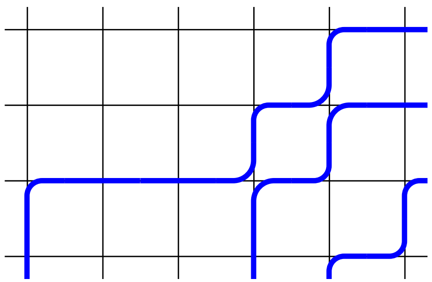
Osculating Path – ASM Correspondence



$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	0	0	0	1	-1

gives a standard bijection between OP(a, b, α, β) and ASM(a, b, α, β)

e.g.



$$\longrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Known Enumeration Formulae

- Standard ASMs: $|\mathrm{OP}(n,n,[n],[n])| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$ *(Zeilberger 1996, Kuperberg 1996)*

- Refined ASM: $|\mathrm{OP}(n,n+1,[n],[n+1]\backslash\{n+1-m\})| = \frac{(2n-m)!\,(n+m)!}{n!\,m!\,(n-m)!} \prod_{i=1}^{n} \frac{(3i-2)!}{(n+i)!}$

  (Zeilberger 1996, Fischer 2007)

- Related case: $|\mathrm{OP}(n,n+m,[n],[n-1]\cup\{n+m\})| =$

  $$\frac{1}{(n-1)!\,m!} \prod_{i=0}^{n-2} \frac{(3i+1)!}{(n+i)!} \sum_{i=0}^{n-1} \frac{(2n-2-i)!\,(n-1+i)!\,(m+i)!}{i!^2\,(n-1-i)!}$$

  (Fischer 2007)

- Vertically Symmetric ASMs: $|\mathrm{OP}(n,2n-1,[n],\{1,3,\ldots,2n-1\})| = \prod_{i=1}^{n} \frac{(6i-2)!}{(2n+2i)!}$

  (Kuperberg 2002)

- Horizontally and Vertically Symmetric ASMs:

  $$|\mathrm{OP}(n,n,\{1,3,\ldots,2\lceil\tfrac{n}{2}\rceil-1\},\{1,3,\ldots,2\lceil\tfrac{n}{2}\rceil-1\})| = \frac{(\lfloor\frac{3n}{2}\rfloor+1)!}{3^{\lfloor\frac{n}{2}\rfloor}\,(2n+1)!\,\lfloor\frac{n}{2}\rfloor!} \prod_{i=1}^{n} \frac{(3i)!}{(n+i)!}$$

  (Okada 2006)

Partitions / Young diagrams

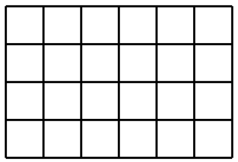
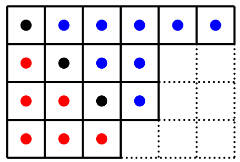
For external parameters a, b, α and β, define the partition / Young diagram

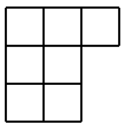
$$\lambda_{a,b,\alpha,\beta} := [a] \times [b] \setminus (b-\beta_1, \ldots, b-\beta_r \,|\, a-\alpha_1, \ldots, a-\alpha_r)$$

(using complement and Frobenius notation)

e.g. $\lambda_{4,6,\{1,2,3\},\{1,4,5\}} = [4] \times [6] \setminus (6-1, 6-4, 6-5 \,|\, 4-1, 4-2, 4-3)$

$= [4] \times [6] \setminus (5, 2, 1 \,|\, 3, 2, 1)$

$=$  $\setminus$ 

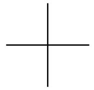
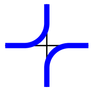
$=$  $= (3, 2, 2)$

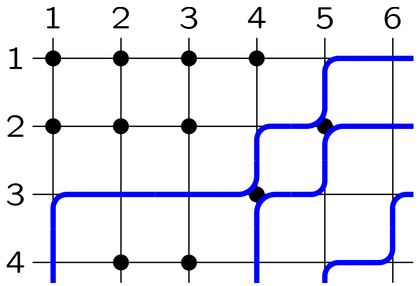
ASM cases:

	$a,\ b,\ \alpha,\ \beta$	$\lambda_{a,b,\alpha,\beta}$
standard	$n,\ n,\ [n],\ [n]$	$\emptyset$
refined	$n,\ n{+}1,\ [n],\ [n{+}1]\setminus\{n{+}1{-}m\}$	$(m)^t$
related	$n,\ n{+}m,\ [n],\ [n{-}1]\cup\{n{+}m\}$	(m)
ver. sym.	$n,\ 2n{-}1,\ [n],\ \{1,3,\dots,2n{-}1\}$	$(n{-}1,n{-}2,\dots,1)$
hor. & ver. sym.	$n,\ n,\ \{1,3,\dots,2\lceil\frac{n}{2}\rceil{-}1\},\ \{1,3,\dots,2\lceil\frac{n}{2}\rceil{-}1\}$	$(n{-}1,n{-}2,\dots,1)$

Vacancies and Osculations

For path tuple $P \in \mathrm{OP}(a, b, \alpha, \beta)$ define

- *vacancies*: points of rectangle through which no path passes, i.e. 
- *osculations*: points of rectangle through which two paths pass, i.e. 

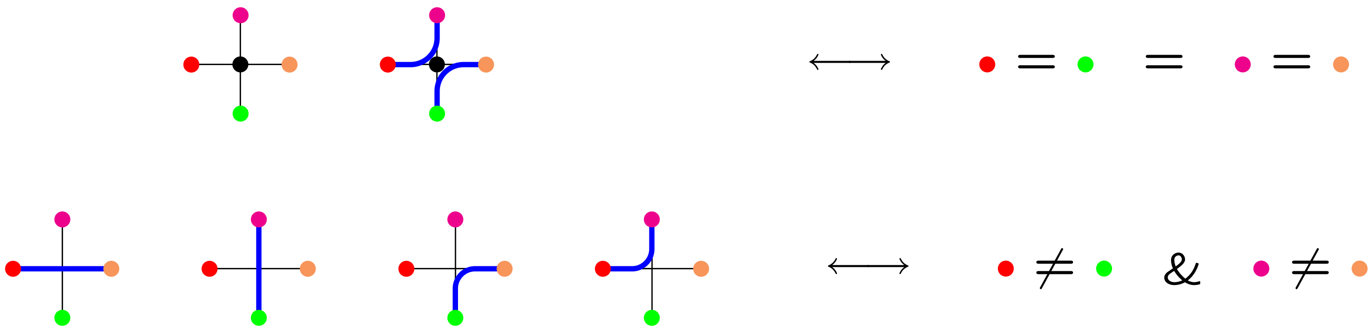
e.g. 

vacancies: $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, $(2,2)$, $(2,3)$, $(4,2)$, $(4,3)$

osculations: $(2,5)$, $(3,4)$

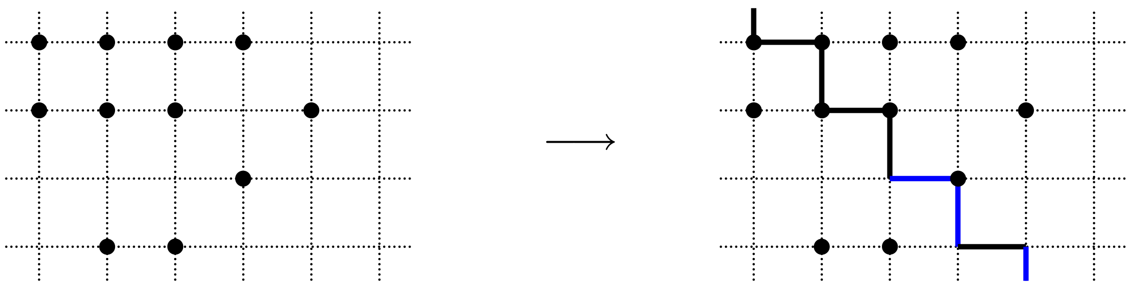
Lemma: Each $P \in \mathrm{OP}(a, b, \alpha, \beta)$ is uniquely determined by its vacancies and osculations

Proof:



$\Rightarrow$ can place path segments successively, moving diagonally downward and rightward from upper and left boundaries of rectangle

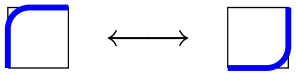
e.g.



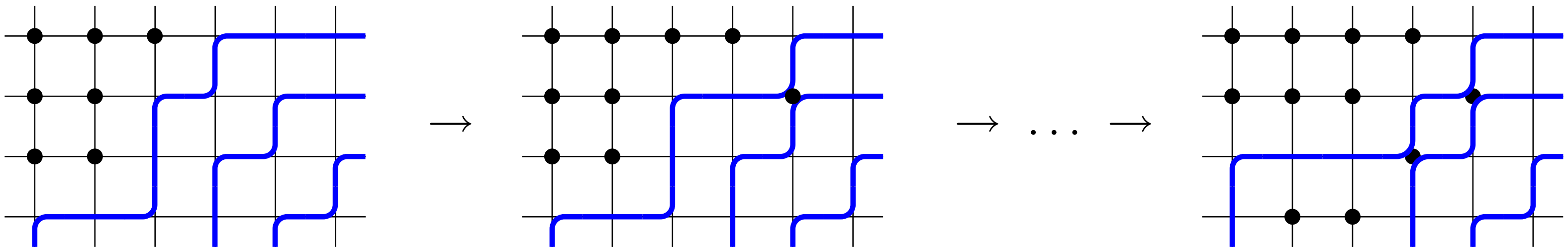
etc.

Lemma: (number of vacancies in P) – (number of osculations in P) $= |\lambda_{a,b,\alpha,\beta}|$ for each $P \in \mathrm{OP}(a, b, \alpha, \beta)$

Proof:

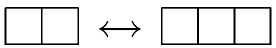
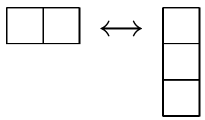
- Start with nonintersecting path tuple with external parameters a, b, α and β, and vacancies forming $\lambda_{a,b,\alpha,\beta}$ in top left corner
- Obtain P by flips 
- Each flip: moves a vacancy/osculation
  or creates a vacancy and an osculation
  or removes a vacancy and an osculation

e.g.

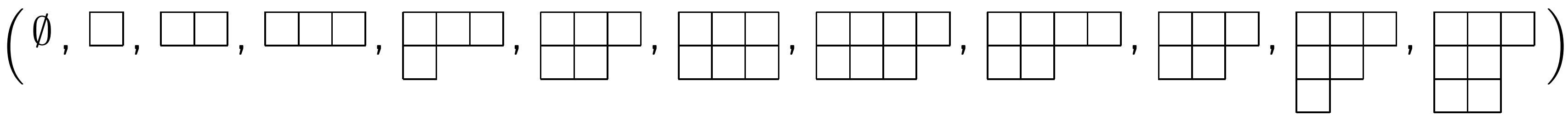


Oscillating Tableaux

For a Young diagram λ and nonnegative integer l, let $\mathrm{OT}(\lambda, l)$ be the set of *oscillating tableaux* of *shape* λ and *length* l, i.e. sequences of $l+1$ Young diagrams in which

- first diagram is $\emptyset$
- last diagram is λ
- successive diagrams differ by the addition or deletion of a single square
  e.g. , but not , allowed

e.g.



$\in \mathrm{OT}((3,2,2),11)$

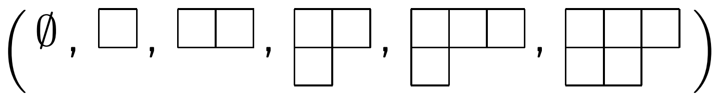
Theorem: $|\mathrm{OT}(\lambda, l)| = \binom{l}{|\lambda|} (l - |\lambda| - 1)!!\, f^\lambda$ *(Sundaram 1986)*

where $f^\lambda =$ number of standard Young tableaux of shape λ

Proof: Bijection between $\mathrm{OT}(\lambda, l)$ and certain pairs
(matching, standard Young tableau)

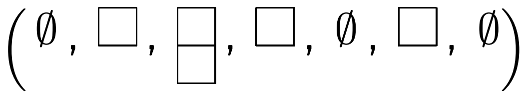
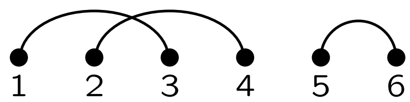
Special cases:

- $\mathrm{OT}(\lambda, |\lambda|) \longleftrightarrow \{\text{standard Young tableaux of shape } \lambda\}$

  e.g.  $\longleftrightarrow$

1	2	4
3	5	

- $\mathrm{OT}(\emptyset, l) \longleftrightarrow \{\text{matchings on } 1, 2, \ldots, l\}$ (l even)

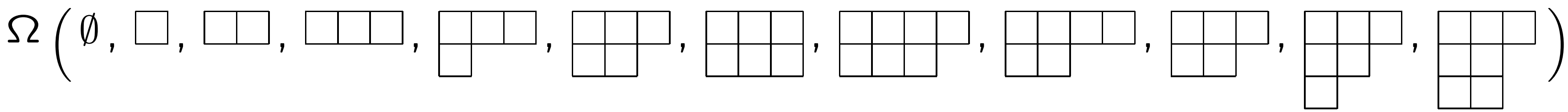
  e.g.  $\longleftrightarrow$ 

Define the *profile* of an oscillating tableau $\eta = (\eta_0, \eta_1, \ldots, \eta_l)$ as

$$\Omega(\eta) := (j_1 - i_1, \ldots, j_l - i_l)$$

where (i_k, j_k) = position of square by which η_k differs from η_{k-1}

e.g.



$$= (0,\ 1,\ 2,\ -1,\ 0,\ 1,\ 3,\ 1,\ 3,\ -2,\ -1)$$

- Each oscillating tableau is uniquely determined by its profile

Generalized Oscillating Tableaux

For a positive integer n and integer q, define the set of *generalized oscillating tableaux* $\mathrm{GOT}(n, q, \lambda, l)$ as the set of pairs $((t_1, \ldots, t_l), \eta)$ in which

- each t_k is an integer between 1 and n
- η is an oscillating tableau of shape λ and length l
- $t_k < t_{k+1}$, or $t_k = t_{k+1}$ and $\Omega(\eta)_k \prec_q \Omega(\eta)_{k+1}$, for each k,

  where $\prec_q$ is the ordering of the integers defined by

  $z \prec_q z'$ if and only if $|z-q| > |z'-q|$ or $z-q = q-z' < 0$

  i.e., $\ldots \prec_q q-2 \prec_q q+2 \prec_q q-1 \prec_q q+1 \prec_q q$

Paths with a Fixed Number of Vacancies and Osculations

Let $\mathsf{OP}(a,b,\alpha,\beta,l) = \{ P \in \mathsf{OP}(a,b,\alpha,\beta) \mid$ (number of vacancies in P) + (number of osculations in P) $= l \}$

Main Result

Theorem: There is a bijection between $\mathrm{OP}(a,b,\alpha,\beta,l)$ and $\mathrm{GOT}(\min(a,b), b-a, \lambda_{a,b,\alpha,\beta}, l)$

Given a path tuple P the corresponding generalized oscillating tableau (t,η) is obtained as:

(1) For each lattice point (i,j), define $L_{i,j} := \begin{cases} \max(i, j+a-b), & a \le b \\ \max(i-a+b, j), & a \ge b \end{cases}$

(2) Order the l vacancies and osculations of P as $(i_1,j_1),\dots,(i_l,j_l)$ with $L_{i_k,j_k} < L_{i_{k+1},j_{k+1}}$, or $L_{i_k,j_k} = L_{i_{k+1},j_{k+1}}$ and $j_k - i_k \prec_{b-a} j_{k+1} - i_{k+1}$

(3) Then $t = (L_{i_1,j_1},\dots,L_{i_l,j_l})$ and η is the oscillating tableau with profile $\Omega(\eta) = (j_1 - i_1,\dots,j_l - i_l)$

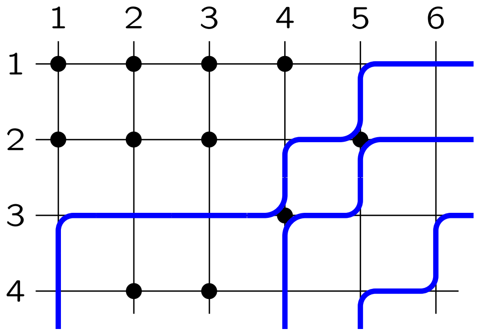
- If (i_k,j_k) is a vacancy, respectively osculation, of P, then η_k is related to η_{k-1} by the addition, respectively deletion, of a square

Corollary: The number of osculating path tuples can be written as a sum over oscillating tableaux

$$|\mathsf{OP}(a,b,\alpha,\beta,l)| = \sum_{\eta \in \mathsf{OT}(\lambda_{a,b,\alpha,\beta},l)} \binom{\min(a,b) + |\mathsf{Asc}_{b-a}(\eta)|}{l}$$

where $\mathsf{Asc}_q(\eta) := \{k \mid \Omega(\eta)_k \prec_q \Omega(\eta)_{k+1}\}$

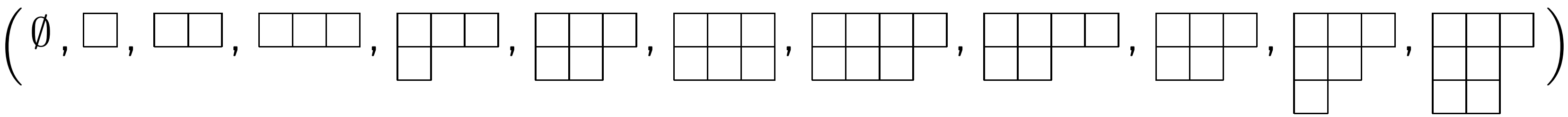
e.g.



(1) $L_{i,j} = \max(i, j-2) \;\rightarrow\; \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 & 3 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 \end{pmatrix}$

(2) Ordered list of vacancies and osculations is $(1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, $(2,2)$, $(2,3)$, $(1,4)$, $(3,4)$, $(2,5)$, $(4,2)$, $(4,3)$

(3) $t = (1,1,1,2,2,2,2,3,3,4,4)$, and $\Omega(\eta) = (0,1,2,-1,0,1,3,1,3,-2,-1)$, so η is the previous example of an oscillating tableau,

$\eta =$ 

Further Example: $\mathrm{OT}(\emptyset, 6)$ $\quad (-2 \prec_0 2 \prec_0 -1 \prec_0 1 \prec_0 0)$

η_0	η_1	η_2	η_3	η_4	η_5	η_6	$\Omega(\eta)_1$	$\Omega(\eta)_2$	$\Omega(\eta)_3$	$\Omega(\eta)_4$	$\Omega(\eta)_5$	$\Omega(\eta)_6$	$\mathrm{Asc}_0(\eta)$
$\emptyset$	$\ydiagram{1}$	$\emptyset$	$\ydiagram{1}$	$\emptyset$	$\ydiagram{1}$	$\emptyset$	0	0	0	0	0	0	$\emptyset$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{2}$	$\ydiagram{1}$	$\emptyset$	$\ydiagram{1}$	$\emptyset$	0	1	1	0	0	0	$\{3\}$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{1}$	$\emptyset$	$\ydiagram{1}$	$\emptyset$	0	–1	–1	0	0	0	$\{3\}$
$\emptyset$	$\ydiagram{1}$	$\emptyset$	$\ydiagram{1}$	$\ydiagram{2}$	$\ydiagram{1}$	$\emptyset$	0	0	0	1	1	0	$\{5\}$
$\emptyset$	$\ydiagram{1}$	$\emptyset$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{1}$	$\emptyset$	0	0	0	–1	–1	0	$\{5\}$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{2}$	$\ydiagram{1}$	$\ydiagram{2}$	$\ydiagram{1}$	$\emptyset$	0	1	1	1	1	0	$\{5\}$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{1}$	$\emptyset$	0	–1	–1	–1	–1	0	$\{5\}$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{2}$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{1}$	$\emptyset$	0	1	1	–1	–1	0	$\{5\}$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{2,1}$	$\ydiagram{1,1}$	$\ydiagram{1}$	$\emptyset$	0	–1	1	1	–1	0	$\{2,5\}$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{1}$	$\ydiagram{2}$	$\ydiagram{1}$	$\emptyset$	0	–1	–1	1	1	0	$\{3,5\}$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{2}$	$\ydiagram{2,1}$	$\ydiagram{1,1}$	$\ydiagram{1}$	$\emptyset$	0	1	–1	1	–1	0	$\{3,5\}$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{2}$	$\ydiagram{2,1}$	$\ydiagram{2}$	$\ydiagram{1}$	$\emptyset$	0	1	–1	–1	1	0	$\{4,5\}$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{2}$	$\ydiagram{3}$	$\ydiagram{2}$	$\ydiagram{1}$	$\emptyset$	0	1	2	2	1	0	$\{4,5\}$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{1,1,1}$	$\ydiagram{1,1}$	$\ydiagram{1}$	$\emptyset$	0	–1	–2	–2	–1	0	$\{4,5\}$
$\emptyset$	$\ydiagram{1}$	$\ydiagram{1,1}$	$\ydiagram{2,1}$	$\ydiagram{2}$	$\ydiagram{1}$	$\emptyset$	0	–1	1	–1	1	0	$\{2,4,5\}$

$\Rightarrow$ Number of $n \times n$ standard ASMs with 3 osculations and 3 vacancies

$$= |\mathsf{OP}(n,n,[n],[n],6)| = \sum_{\eta \in \mathsf{OT}(\emptyset,6)} \binom{n+|\mathsf{Asc}_0(\eta)|}{6}$$

$$= \binom{n}{6} + 7\binom{n+1}{6} + 6\binom{n+2}{6} + \binom{n+3}{6}$$

Part of Proof of Bijection

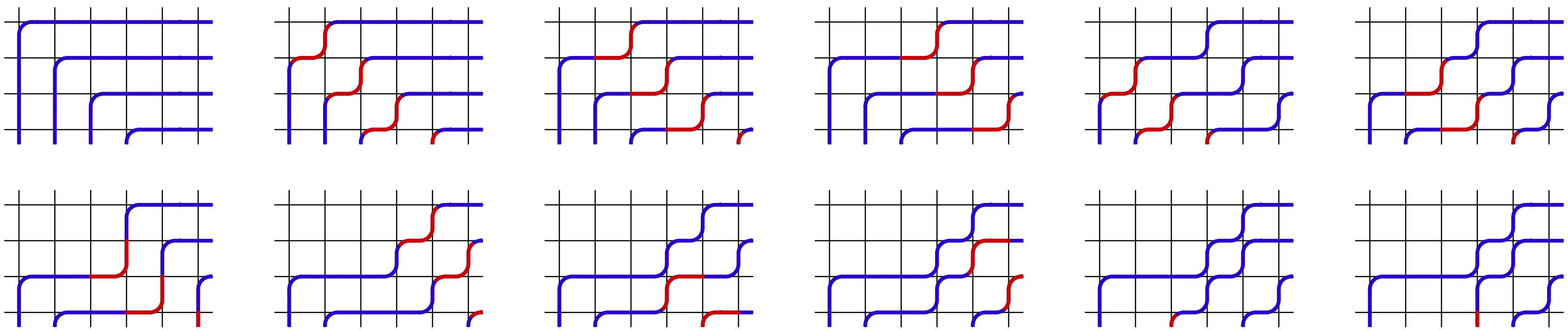
For $P \in \mathrm{OP}(a, b, \alpha, \beta, l)$, create a sequence of $l+1$ path tuples in the a by b rectangle in which

- first tuple has no vacancies or osculations
- each successive tuple has a single additional vacancy or osculation of P, using ordering of bijection

Implies

- successive tuples differ by flips along single diagonal beyond added vac./osc.
- successive boundary conditions differ at two adjacent boundary points
- Young diagrams of successive boundary conditions differ by a single square (with addition for a vacancy, deletion for an osculation)
- first & last Young diagrams are $\emptyset$ & $\lambda_{a,b,\alpha,\beta}$, so sequence is in $\mathrm{OT}(\lambda_{a,b,\alpha,\beta}, l)$

e.g.



Possible Further Work

- Use the osculating paths – oscillating tableaux bijection, other known bijections and Lindström–Gessel–Viennot theorem to obtain determinantal enumeration formulae, e.g.

$$\sum_{l=0}^{n(n-1)/2} |\mathrm{OP}(n,n,[n],[n],2l)|\, x^l = \det_{0\le i,j\le n-1}\left(\binom{i+j}{i} - x^i\,\delta_{i,j+1}\right)$$

- Study osculating paths with other external configurations

- Find representation theoretic interpretation of generalized oscillating tableaux