Polytopes and Alternating Sign Matrices

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Outline

- Define the Birkhoff polytope $\mathcal{B}_n$ and alternating sign matrix polytope $\mathcal{A}_n$.
- Give counterparts for $\mathcal{A}_n$ of some standard results for $\mathcal{B}_n$, e.g. characterization of faces and vertices.
- Consider integer points of dilates of $\mathcal{B}_n$ and $\mathcal{A}_n$.
- Identify connection between $\mathcal{A}_n$ and higher spin vertex models with domain wall boundary conditions.

Some References

- R Behrend & V Knight *Higher spin alternating sign matrices*
  Electronic J. Combinatorics 14 (2007) #R83
- J Striker *The alternating sign matrix polytope*
  Electronic J. Combinatorics 16 (2009) #R41
- R Behrend & V Knight *Partial sum transportation polytopes*
  In preparation
- R Behrend *Fractional perfect b-matching polytopes*
  In preparation

Polytopes

- *Polytope*: a bounded intersection of finitely-many closed halfspaces (and hyperplanes) in $\mathbb{R}^d$, i.e. a bounded set $\{x \in \mathbb{R}^d \mid a_1 . x \leq b_1, \ldots, a_k . x \leq b_k\}$, for some fixed $a_1, \ldots, a_k \in \mathbb{R}^d$, $b_1, \ldots, b_k \in \mathbb{R}$

  Equivalently (Minkowski, Weyl) a convex hull of finitely-many points in $\mathbb{R}^d$, i.e. a set $\{\lambda_1 v_1 + \ldots + \lambda_m v_m \mid \lambda_1, \ldots, \lambda_m \geq 0,\ \lambda_1 + \ldots + \lambda_m = 1\}$, for some fixed $v_1, \ldots, v_m \in \mathbb{R}^d$

- *Face of polytope $\mathcal{P}$*: intersection of $\mathcal{P}$ with any hyperplane H for which $\mathcal{P}$ is contained on one side of H.

  Obtained by changing some inequalities $a_i . x \leq b_i$ to equalities $a_i . x = b_i$.

- *Dimension of face F of polytope*: $\dim F = \dim\{\lambda x_1 + (1-\lambda) x_2 \mid x_1, x_2 \in F,\ \lambda \in \mathbb{R}\}$

- *Vertex of polytope $\mathcal{P}$*: point of $\mathcal{P}$ which does not lie in the interior of any line segment in $\mathcal{P}$.
  Corresponds to face of dimension 0.

Birkhoff and Alternating Sign Matrix Polytopes

- *Birkhoff polytope (polytope of doubly stochastic matrices)*:

$$\mathcal{B}_n := \left\{ \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} \;\middle|\; x_{ij} \geq 0,\ \sum_{j'=1}^{n} x_{ij'} = \sum_{i'=1}^{n} x_{i'j} = 1,\ \text{for all } i,j = 1,\dots,n \right\}$$

$$= \left\{ n \times n \text{ real matrices} \;\middle|\; \begin{array}{l} \bullet \text{ each entry is nonnegative} \\ \bullet \text{ each complete row \& column sum is 1} \end{array} \right\}$$

- *Alternating sign matrix polytope:*

$$\mathcal{A}_n := \left\{ \begin{pmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \dots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} \;\middle|\; \begin{array}{l} \sum_{j'=1}^{j} x_{ij'} \geq 0,\ \sum_{j'=j}^{n} x_{ij'} \geq 0,\ \sum_{i'=1}^{i} x_{i'j} \geq 0,\ \sum_{i'=i}^{n} x_{i'j} \geq 0, \\ \sum_{j'=1}^{n} x_{ij'} = \sum_{i'=1}^{n} x_{i'j} = 1,\ \text{for all } i,j = 1,\dots,n \end{array} \right\}$$

$$= \left\{ n \times n \text{ real matrices} \;\middle|\; \begin{array}{l} \bullet \text{ each partial row \& column sum extending from} \\ \quad \text{each end of the row or column is nonnegative} \\ \bullet \text{ each complete row \& column sum is 1} \end{array} \right\}$$

Properties:
- $\mathcal{B}_n \subset \mathcal{A}_n$
- $x \in \mathcal{B}_n \Rightarrow 0 \leq x_{ij} \leq 1$
- $\dim \mathcal{B}_n = \dim \mathcal{A}_n = (n-1)^2$
- $x \in \mathcal{A}_n \Rightarrow -1 \leq x_{ij} \leq 1$

- $\mathcal{B}_n$ contains all $n \times n$ permutation matrices

- $\mathcal{A}_n$ contains all $n \times n$ alternating sign matrices
  (matrices with entries 0, 1 & -1 in which the nonzero entries alternate in sign along each row & column, and each row & column sum is 1)

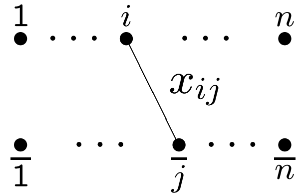
e.g. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} .6 & .1 & .3 \\ 0 & .8 & .2 \\ .4 & .1 & .5 \end{pmatrix} \in \mathcal{B}_3 \subset \mathcal{A}_3$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} .3 & 0 & .6 & .1 \\ .2 & .5 & -.6 & .9 \\ .5 & -.5 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathcal{A}_4$$

Graph Versions of $\mathcal{B}_n$ and $\mathcal{A}_n$

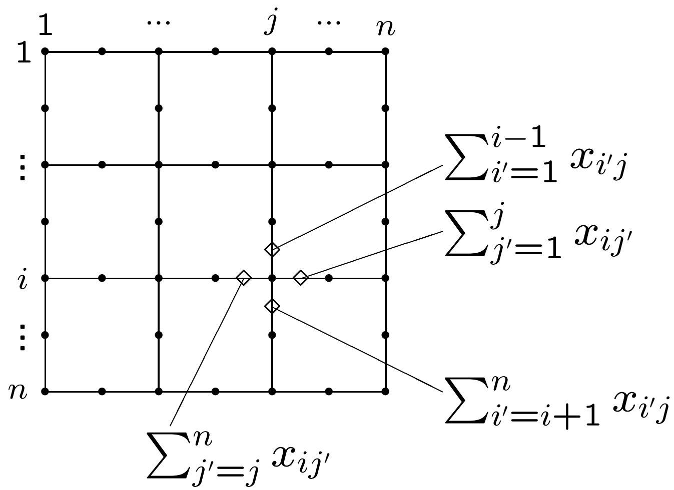
- $\mathcal{B}_n \leftrightarrow \left\{ \begin{array}{l} \text{labelings of edges of} \\ \text{complete bipartite graph } K_{n,n} \end{array} \middle| \begin{array}{l} \bullet \text{ each label is real \& nonnegative} \\ \bullet \text{ sum of labels over all edges} \\ \quad \text{incident to any vertex is 1} \end{array} \right\}$

$$\begin{pmatrix} & \vdots & \\ \dots & x_{ij} & \dots \\ & \vdots & \end{pmatrix} \quad \leftrightarrow$$



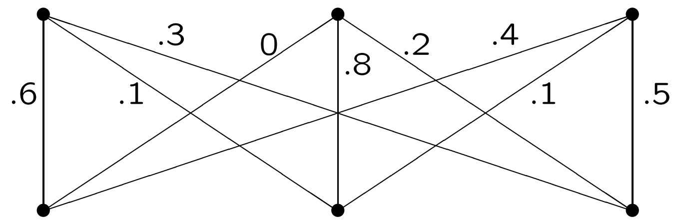
- $\mathcal{A}_n \leftrightarrow \left\{ \begin{array}{l} \text{labelings of edges of } n\times n \text{ grid} \\ \text{graph with intermediate vertices} \end{array} \middle| \begin{array}{l} \bullet \text{ each label is real \& nonnegative} \\ \bullet \text{ sum of labels over all edges} \\ \quad \text{incident to a vertex is 1 (for each} \\ \quad \text{intermediate \& vertical boundary} \\ \quad \text{vertex) or 2 (all other vertices)} \end{array} \right\}$

$$\begin{pmatrix} & \vdots & \\ \dots & x_{ij} & \dots \\ & \vdots & \end{pmatrix} \quad \leftrightarrow$$

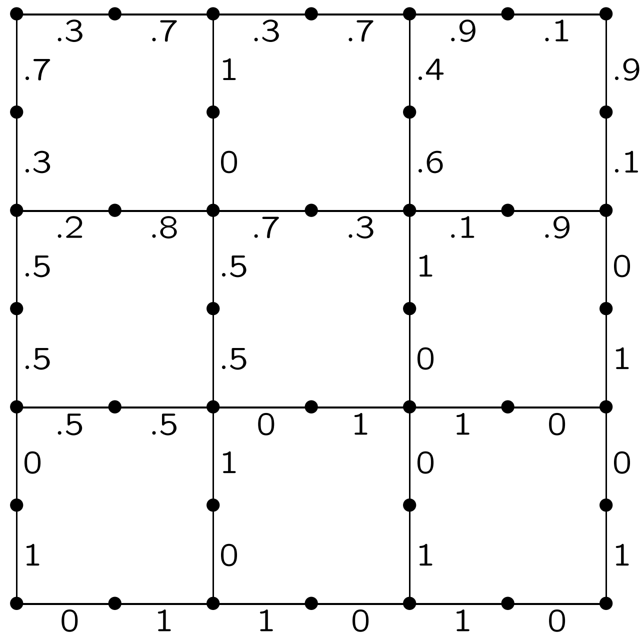


e.g.

$$\begin{pmatrix} .6 & .1 & .3 \\ 0 & .8 & .2 \\ .4 & .1 & .5 \end{pmatrix} \in \mathcal{B}_3 \quad \leftrightarrow$$



$$\begin{pmatrix} .3 & 0 & .6 & .1 \\ .2 & .5 & -.6 & .9 \\ .5 & -.5 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathcal{A}_4 \quad \leftrightarrow$$



Faces of $\mathcal{B}_n$ and $\mathcal{A}_n$

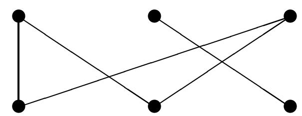
- *Graph G_F of a face F of $\mathcal{B}_n$*: obtained from $K_{n,n}$ by deleting each edge corresponding to an entry which is 0 for all elements of F

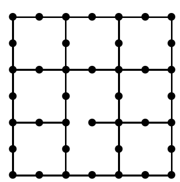
- *Graph G_F of a face F of $\mathcal{A}_n$*: obtained from $n \times n$ grid graph with intermediate vertices by deleting each edge corresponding to a partial row or column sum which is 0 for all elements of F

- *Lattice isomorphism*: each mapping from faces to graphs is an isomorphism between the lattice of faces and a lattice of certain 'elementary' spanning subgraphs

- *Dimension of face F with graph G_F*: using certain results for polytopes & graphs,

$$\begin{aligned}\dim F &= \dim(\text{kernel of incidence matrix of } G_F)\\ &= (\#\text{ of edges in } G_F) - (\#\text{ of vertices in } G_F) + (\#\text{ of components in } G_F)\\ &= \begin{cases}(\#\text{ of edges in } G_F) - 2n + (\#\text{ of components in } G_F), & \text{for } \mathcal{B}_n\\ (\#\text{ of bounded faces in } G_F), & \text{for } \mathcal{A}_n \text{ (using Euler planar graph formula)}\end{cases}\end{aligned}$$

e.g. $\{x \in \mathcal{B}_3 \mid x_{13} = x_{21} = x_{22} = x_{33} = 0\} \;\leftrightarrow$  dim $= 5-6+2 = 1$

$\{x \in \mathcal{A}_4 \mid x_{31}+x_{32} = 0\} \;\leftrightarrow$  dim $= 8$

Number of faces of $\mathcal{B}_3$:

dimension	-1	0	1	2	3	4
number	1	6	15	18	9	1

Number of faces of $\mathcal{A}_3$:

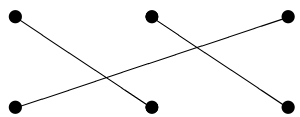
dimension	-1	0	1	2	3	4
number	1	7	17	18	8	1

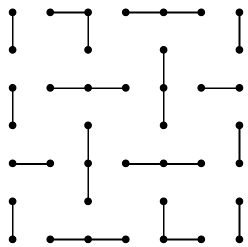
Vertices of $\mathcal{B}_n$ and $\mathcal{A}_n$

- *Graph G_x of $x \in \mathcal{B}_n$*: obtained from $K_{n,n}$ by deleting each edge corresponding to a zero entry of x
- *Graph G_x of $x \in \mathcal{A}_n$*: obtained from $n \times n$ grid graph with intermediate vertices by deleting each edge corresponding to a zero partial row or column sum of x

Characterization of vertices: setting $\dim(F) = 0$ in previous formula gives
x is a vertex of $\mathcal{B}_n$ or $\mathcal{A}_n$ if and only if G_x is a forest

Implies:
- vertices of $\mathcal{B}_n$ are all $n \times n$ permutation matrices *(Birkhoff 1946)*
- vertices of $\mathcal{A}_n$ are all $n \times n$ ASMs *(RB, Knight 2007; Striker 2009)*

e.g. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \leftrightarrow$ 

$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \leftrightarrow$ 

- graph of permutation matrix $\leftrightarrow$ perfect matching of $K_{n,n}$
- graph of alternating sign matrix $\leftrightarrow$ modified configuration of 6-vertex model on $n \times n$ grid with domain-wall boundary conditions

Generalizations of $\mathcal{B}_n$ and $\mathcal{A}_n$

For positive $r_1, \ldots, r_m$, $c_1, \ldots, c_n$ with $r_1 + \ldots + r_m = c_1 + \ldots + c_n$:

- *Transportation polytope*

$$\mathcal{T}(r,c) := \left\{ m \times n \text{ real matrices} \;\middle|\; \begin{array}{l} \bullet \text{ each entry is nonnegative} \\ \bullet \text{ row sum } i \text{ is } r_i \text{, for each } i = 1, \ldots, m \\ \bullet \text{ column sum } j \text{ is } c_j \text{, for each } j = 1, \ldots, n \end{array} \right\}$$

- *Partial sum transportation polytope*

$$\mathcal{P}(r,c) := \left\{ m \times n \text{ real matrices} \;\middle|\; \begin{array}{l} \bullet \text{ each partial row \& column sum extending from} \\ \quad \text{each end of the row or column is nonnegative} \\ \bullet \text{ row sum } i \text{ is } r_i \text{, for each } i = 1, \ldots, m \\ \bullet \text{ column sum } j \text{ is } c_j \text{, for each } j = 1, \ldots, n \end{array} \right\}$$

Previous results for $\mathcal{B}_n$ and $\mathcal{A}_n$ can be generalized

Integer Points of Dilates of $\mathcal{B}_n$ and $\mathcal{A}_n$

For a nonnegative integer r:

- *Semimagic squares*

  $$\mathrm{SMS}(n,r) := \left\{ n\times n \text{ integer matrices} \;\middle|\; \begin{array}{l} \bullet \text{ each entry is nonnegative} \\ \bullet \text{ each complete row \& column sum is } r \end{array} \right\}$$
  $$= \mathbb{Z}^{n^2} \cap (r\mathcal{B}^n)$$

- *Generalized alternating sign matrices*

  $\mathrm{ASM}(n,r) :=$
  $$\left\{ n\times n \text{ integer matrices} \;\middle|\; \begin{array}{l} \bullet \text{ each partial row \& column sum extending from} \\ \quad \text{each end of the row or column is nonnegative} \\ \bullet \text{ each complete row \& column sum is } r \end{array} \right\}$$
  $$= \mathbb{Z}^{n^2} \cap (r\mathcal{A}^n)$$

e.g. $\begin{pmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} \in \mathrm{SMS}(4,3) \subset \mathrm{ASM}(4,3)$ $\quad \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -2 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{ASM}(5,2)$

- $\{$sums of r $n\times n$ permutation matrices$\}$ $\subset \mathrm{SMS}(n,r)$,
  $\{$sums of r $n\times n$ standard alternating sign matrices$\}$ $\subset \mathrm{ASM}(n,r)$
- Reverse containments also, less trivially, valid

Cardinalities of SMS(n,r) and ASM(n,r)

$\lvert\mathrm{SMS}(n,r)\rvert$	$r=0$	1	2	3	4
$n=1$	1	1	1	1	1
2	1	2	3	4	5
3	1	6	21	55	120
4	1	24	282	2008	10147
5	1	120	6210	153040	2224955
6	1	720	202410	20933840	1047649905

$\lvert\mathrm{ASM}(n,r)\rvert$	$r=0$	1	2	3	4
$n=1$	1	1	1	1	1
2	1	2	3	4	5
3	1	7	26	70	155
4	1	42	628	5102	28005
5	1	429	41784	1507128	28226084
6	1	7436	7517457	1749710096	152363972022

- $n=1$: single 1×1 matrix (r)
- $n=2$: $\begin{pmatrix} i & r-i \\ r-i & i \end{pmatrix}$ for $i=0,1,\ldots,r$

- Ehrhart theory implies $|\mathrm{SMS}(n,r)|$ & $|\mathrm{ASM}(n,r)|$ for fixed n are polynomials in r of the form $\displaystyle\sum_{k=n-1}^{(n-1)^2} c_{nk}\binom{r+k}{(n-1)^2}$, where c_{nk} are nonnegative integers.

e.g. $|\mathrm{SMS}(3,r)| = \binom{r+2}{4} + \binom{r+3}{4} + \binom{r+4}{4}$

$|\mathrm{ASM}(3,r)| = \binom{r+2}{4} + 2\binom{r+3}{4} + \binom{r+4}{4}$

$|\mathrm{SMS}(4,r)| = \binom{r+3}{9} + 14\binom{r+4}{9} + 87\binom{r+5}{9} + 148\binom{r+6}{9} + 87\binom{r+7}{9} + 14\binom{r+8}{9} + \binom{r+9}{9}$

$|\mathrm{ASM}(4,r)| = 3\binom{r+3}{9} + 80\binom{r+4}{9} + 415\binom{r+5}{9} + 592\binom{r+6}{9} + 253\binom{r+7}{9} + 32\binom{r+8}{9} + \binom{r+9}{9}$

- Known formulae for fixed r:

$$|\mathrm{SMS}(n,0)| = |\mathrm{ASM}(n,0)| = 1 \quad \text{(single } n\times n \text{ zero matrix)}$$

$$|\mathrm{SMS}(n,1)| = n! \quad (n\times n \text{ permutation matrices})$$

$$|\mathrm{ASM}(n,1)| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} \quad (n\times n \text{ standard alternating sign matrices})$$

$$|\mathrm{SMS}(n,2)| = \sum_{i=0}^{n} \frac{(n!)^2\,(2i)!}{2^{n+i}\,(i!)^2\,(n-i)!}$$

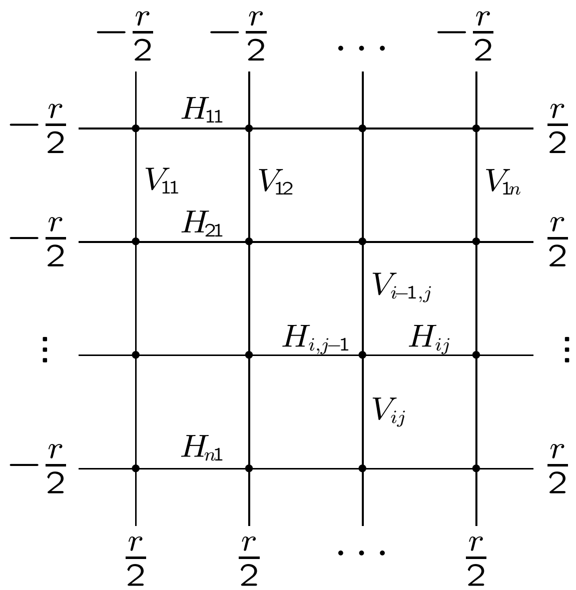
$$|\mathrm{SMS}(n,3)| = \sum_{\substack{\text{nonneg. integers} \\ i,j,k \text{ with } i+j+k=n}} \frac{(n!)^2\,(3i+j)!}{2^{2i+j}\,3^{2i+k}\,(i!)^2\,j!\,k!}$$

Higher Spin Vertex Models with Domain-Wall Boundary Conditions

Configuration of spin-$\frac{r}{2}$ vertex model on $n\times n$ grid with generalized DWBC:

assignment of H_{ij} to horizontal edges and V_{ij} to vertical edges with

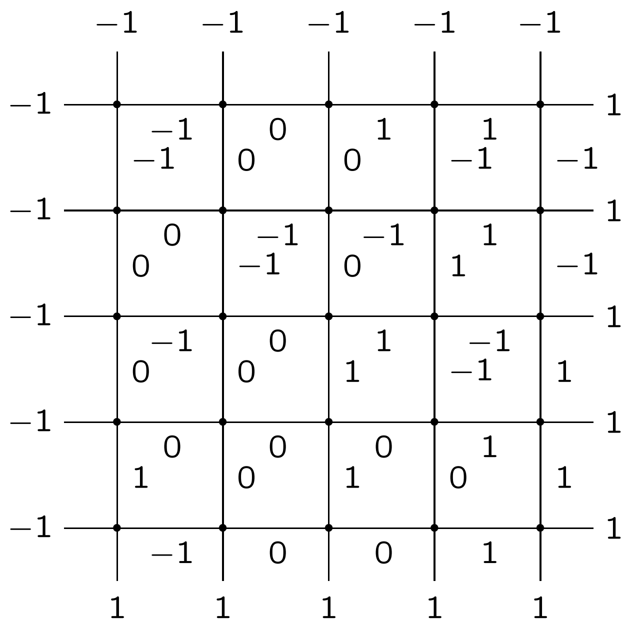
- $H_{ij}, V_{ij} \in \{-\frac{r}{2}, -\frac{r}{2}+1, \ldots, \frac{r}{2}-1, \frac{r}{2}\}$
- $H_{i0} = V_{0j} = -\frac{r}{2}$ (left & upper boundaries)
- $H_{in} = V_{nj} = \frac{r}{2}$ (right & lower boundaries)
- $H_{i,j-1} + V_{ij} = V_{i-1,j} + H_{ij}$ ('ice condition')



- ASM(n,r) & {configurations of spin-$\frac{r}{2}$ vertex model on $n\times n$ grid with DWBC} are in bijection, with $A\in$ ASM(n,r) & corresponding configuration (H,V) related by:

$$H_{ij}=-\frac{r}{2}+\sum_{j'=1}^{j}A_{ij'}\,,\qquad V_{ij}=-\frac{r}{2}+\sum_{i'=1}^{i}A_{i'j}\,,\qquad A_{ij}\;=\;H_{ij}-H_{i,j-1}\;=\;V_{ij}-V_{i-1,j}$$

e.g. $\begin{pmatrix}0&1&1&0&0\\1&-1&0&2&0\\0&1&1&-2&2\\1&0&0&1&0\\0&1&0&1&0\end{pmatrix}\in$ ASM$(5,2)$ $\longleftrightarrow$



Integrable Boltzmann weights can be assigned to vertex configurations.

For $r = 1$ can use integrable six-vertex model weights

$$w_{\lambda,z}\left(\begin{smallmatrix} & -\frac{1}{2} & \\ -\frac{1}{2} & + & -\frac{1}{2} \\ & -\frac{1}{2} & \end{smallmatrix}\right) = w_{\lambda,z}\left(\begin{smallmatrix} & \frac{1}{2} & \\ \frac{1}{2} & + & \frac{1}{2} \\ & \frac{1}{2} & \end{smallmatrix}\right) = \sin(z+\lambda), \quad w_{\lambda,z}\left(\begin{smallmatrix} & \frac{1}{2} & \\ -\frac{1}{2} & + & -\frac{1}{2} \\ & \frac{1}{2} & \end{smallmatrix}\right) = w_{\lambda,z}\left(\begin{smallmatrix} & -\frac{1}{2} & \\ \frac{1}{2} & + & \frac{1}{2} \\ & -\frac{1}{2} & \end{smallmatrix}\right) = \sin(z-\lambda),$$

$$w_{\lambda,z}\left(\begin{smallmatrix} & \frac{1}{2} & \\ \frac{1}{2} & + & -\frac{1}{2} \\ & -\frac{1}{2} & \end{smallmatrix}\right) = w_{\lambda,z}\left(\begin{smallmatrix} & -\frac{1}{2} & \\ -\frac{1}{2} & + & \frac{1}{2} \\ & \frac{1}{2} & \end{smallmatrix}\right) = \sin(2\lambda)$$

- $z =$ *spectral parameter* $\qquad \lambda =$ *crossing parameter*
- *Yang-Baxter equation* satisfied
- Related to spin-$\frac{1}{2}$, i.e. highest weight 1, irreducible representation of $sl(2,\mathbb{C})$
- *Partition function* with spectral parameter $x_i - y_j$ at vertex (i,j) (for some $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n)$) given by *Izergin-Korepin determinant*

$$Z(n,\lambda,x,y) \;\equiv \sum_{\substack{\text{configurations}\\(H,V)}} \;\prod_{i,j=1}^{n} w_{\lambda,x_i-y_j}\left(\begin{smallmatrix} & V_{i-1,j} & \\ H_{i,j-1} & + & H_{ij} \\ & V_{ij} & \end{smallmatrix}\right)$$

$$= \frac{\sin(2\lambda)^n \prod_{i,j=1}^{n} \sin(x_i-y_j-\lambda)\sin(x_i-y_j+\lambda)}{\prod_{1\le i<j\le n} \sin(x_i-y_j)\sin(y_j-y_i)} \det_{1\le i,j\le n}\left(\frac{1}{\sin(x_i-y_j-\lambda)\sin(x_i-y_j+\lambda)}\right)$$

- Can be used in derivation of formula for $|\mathrm{ASM}(n,1)|$ *(Kuperberg 1996)*

Boltzmann weights for $r > 1$ obtained by applying *fusion* to $r = 1$ weights:

- $w_{r,\lambda,z}\left(h' \overset{v'}{\underset{v}{+}} h\right)$ given in terms of $r \times r$ block of spin-$\frac{1}{2}$ weights with spectral parameter of (k,l) entry of block $= z + 2(l-k)\lambda$ and certain eigenvector entries P of *fusion projectors* applied at boundaries *(Kulish, Reshetikhin, Sklyanin 1981)*
- Also satisfy Yang-Baxter equation
- Related to spin-$\frac{r}{2}$, i.e. highest weight r, irreducible representation of $sl(2,\mathbb{C})$

e.g. $r = 2$:

$$w_{2,\lambda,z}\left(h' \overset{v'}{\underset{v}{+}} h\right) =$$

$$\frac{1}{\sin(z-\lambda)\sin(z+\lambda)} \sum_{a,b} P(2,h')_{a_{11}a_{21}} \begin{array}{c} P(2,v')_{b_{11}b_{12}} \\ w_{\lambda,z}\left(a_{11} \overset{b_{11}}{\underset{b_{21}}{+}} a_{12}\right) \; w_{\lambda,z+2\lambda}\left(a_{12} \overset{b_{12}}{\underset{b_{22}}{+}} a_{13}\right) \\ w_{\lambda,z-2\lambda}\left(a_{21} \overset{b_{21}}{\underset{b_{31}}{+}} a_{22}\right) \; w_{\lambda,z}\left(a_{22} \overset{b_{22}}{\underset{b_{32}}{+}} a_{23}\right) \\ P(2,v)_{b_{31}b_{32}} \end{array} P(2,h)_{a_{13}a_{23}}$$

which gives

$$w_{2,\lambda,z}\left(\begin{smallmatrix} & -1 & \\ -1 & + & -1 \\ & -1 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & 1 & \\ 1 & + & 1 \\ & 1 & \end{smallmatrix}\right) = \sin(z+\lambda)\sin(z+3\lambda)$$

$$w_{2,\lambda,z}\left(\begin{smallmatrix} & 1 & \\ -1 & + & -1 \\ & 1 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & -1 & \\ 1 & + & 1 \\ & -1 & \end{smallmatrix}\right) = \sin(z-\lambda)\sin(z-3\lambda)$$

$$w_{2,\lambda,z}\left(\begin{smallmatrix} & 0 & \\ -1 & + & -1 \\ & 0 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & -1 & \\ 0 & + & 0 \\ & -1 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & 1 & \\ 0 & + & 0 \\ & 1 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & 0 & \\ 1 & + & 1 \\ & 0 & \end{smallmatrix}\right) = \sin(z-\lambda)\sin(z+\lambda)$$

$$w_{2,\lambda,z}\left(\begin{smallmatrix} & -1 & \\ -1 & + & 0 \\ & 0 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & 0 & \\ 0 & + & -1 \\ & -1 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & 0 & \\ 0 & + & 1 \\ & 1 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & 1 & \\ 1 & + & 0 \\ & 0 & \end{smallmatrix}\right) = \sin(4\lambda)\sin(z+\lambda)$$

$$w_{2,\lambda,z}\left(\begin{smallmatrix} & 0 & \\ -1 & + & 0 \\ & 1 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & 1 & \\ 0 & + & -1 \\ & 0 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & -1 & \\ 0 & + & 1 \\ & 0 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & 0 & \\ 1 & + & 0 \\ & -1 & \end{smallmatrix}\right) = \sin(4\lambda)\sin(z-\lambda)$$

$$w_{2,\lambda,z}\left(\begin{smallmatrix} & -1 & \\ -1 & + & 1 \\ & 1 & \end{smallmatrix}\right) = w_{2,\lambda,z}\left(\begin{smallmatrix} & 1 & \\ 1 & + & -1 \\ & -1 & \end{smallmatrix}\right) = \sin(2\lambda)\sin(4\lambda)$$

$$w_{2,\lambda,z}\left(\begin{smallmatrix} & 0 & \\ 0 & + & 0 \\ & 0 & \end{smallmatrix}\right) = \sin(2\lambda)\sin(4\lambda) + \sin(z-\lambda)\sin(z-\lambda)$$

Partition function for arbitrary r with spectral parameter $x_i - y_j$ at vertex (i,j) given in terms of certain $nr \times nr$ Izergin-Korepin determinant:

$$Z(n,r,\lambda,x,y) \;=\; \ldots \;\; \det M(n,r,\lambda,x,y)$$

$$\text{with } M(n,r,\lambda,x,y)_{(i,k),(j,l)} \;=\; \frac{1}{\sin(x_i - y_j + (2(l-k)-1)\lambda)\,\sin(x_i - y_j + (2(l-k)+1)\lambda)}$$

$$i,j = 1,\ldots n; \quad k,l = 1,\ldots,r$$

(Caradoc, Foda, Kitanine 2006)

Sketch of derivation:

- $Z(n,r,\lambda,x,y)$ involves $n \times n$ block of $r \times r$ blocks of spin-$\frac{1}{2}$ weights
- Fusion projectors satisfy *push-through property* which allows all internal projector eigenvectors to be moved to boundaries, where they give spin-$\frac{1}{2}$ DWBC
- $Z(n,r,\lambda,x,y)$ is then proportional to spin-$\frac{1}{2}$ partition function on $nr \times nr$ grid with DWBC and certain spectral parameter assignments

Generalized Osculating Path Collections

(n, r) *osculating path collection*: nr noncrossing paths on $n \times n$ grid for which

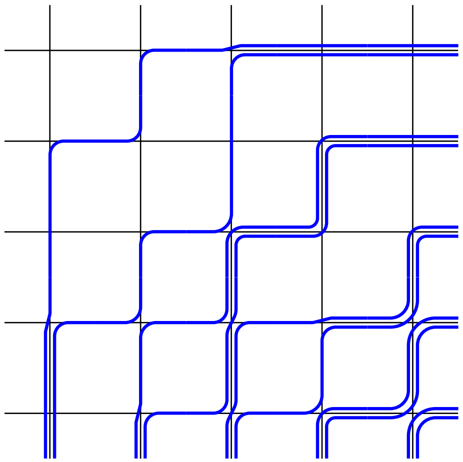
- r paths begin/end at each point in last row/column
- only unit upward & rightward steps allowed
- at most r paths pass along any edge

ASM(n, r) & $\{(n, r)$ osculating path collections$\}$ are in bijection, with corresponding $A \in$ ASM(n, r) & path collection related by:

$$(\#\text{ paths on horizontal edge between }(i,j)\ \&\ (i,j{+}1)) = \sum_{j'=1}^{j} A_{ij'}$$

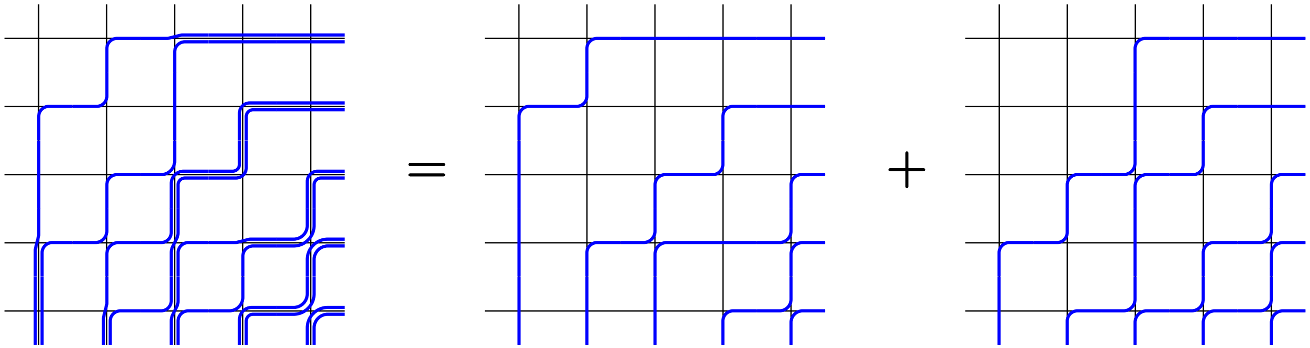
$$(\#\text{ paths on vertical edge between }(i,j)\ \&\ (i{+}1,j)) = \sum_{i'=1}^{i} A_{i'j}$$

e.g.
$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -2 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \in \text{ASM}(5,2) \quad \longleftrightarrow$$



- Separability of paths implies any matrix in ASM(n,r) is a (not necessarily unique) sum of r standard $n \times n$ alternating sign matrices

e.g.



gives

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -2 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Generalized Fully Packed Loop Configurations

(n, r) *fully packed loop configuration*:

Collection of noncrossing paths on $n \times n$ grid for which

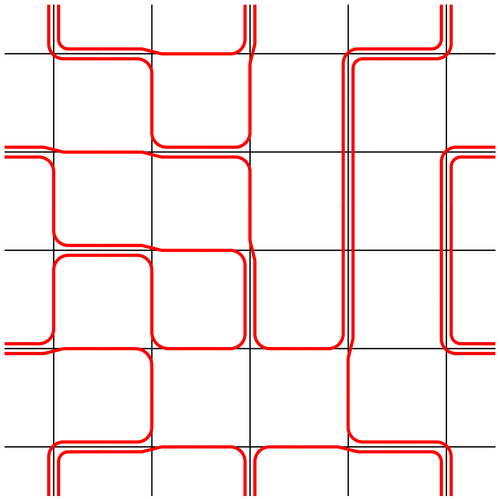
- numbers of path endpoints are alternately 0 & r along boundaries
- only unit upward, downward, leftward & rightward steps allowed
- exactly r path segments pass through each internal vertex

Surjection from fully packed loop configuration to $A \in \text{ASM}(n, r)$ given by

$$\begin{pmatrix} \sum_{j'=1}^{j} A_{ij'}, \ i+j \text{ odd}, \\ \sum_{j'=j+1}^{n} A_{ij'}, \ i+j \text{ even} \end{pmatrix} = (\# \text{ segments on edge between } (i, j) \ \& \ (i, j+1))$$

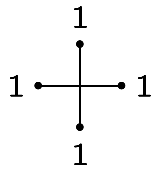
$$\begin{pmatrix} \sum_{i'=i+1}^{n} A_{i'j}, \ i+j \text{ odd}, \\ \sum_{i'=1}^{i} A_{i'j}, \ i+j \text{ even} \end{pmatrix} = (\# \text{ segments on edge between } (i, j) \ \& \ (i+1, j))$$

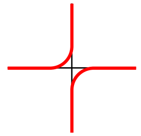
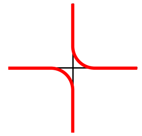
e.g.



$$\longrightarrow \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 0 & 1 & 1 & -2 & 2 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \in \text{ASM}(5, 2)$$

- For $r \geq 2$, several fully packed loop configurations can map to same matrix, since particular numbers of paths on edges can have several connections at vertices

  e.g. for $r = 2$, 

can connect as  or 

  Therefore, # (n, r) fully packed loop configurations
  = weighted sum over ASM(n, r)

- Higher spin link patterns and integrable loop models studied by *Zinn-Justin 2007*

Summary

- The alternating sign matrix polytope $\mathcal{A}_n$ naturally extends the Birkhoff polytope.
  - The faces of $\mathcal{A}_n$ correspond to certain spanning subgraphs of an $n \times n$ grid-type graph.
  - The vertices of $\mathcal{A}_n$ are all $n \times n$ alternating sign matrices.

- The set ASM(n, r) of integer points of the r-th dilate of $\mathcal{A}_n$ is in bijection with the set of configurations of spin-$\frac{r}{2}$ integrable vertex models on an $n \times n$ grid with generalized domain-wall boundary conditions.
  - The partition function of these models can be expressed as an Izergin–Korepin-type determinant.
  - There is also a bijection between certain osculating path collections and ASM(n, r), and a surjection from certain fully packed loop configurations to ASM(n, r).