Proof of the Alternating Sign Matrix and Descending Plane Partition Conjecture

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Full details: RB, P Di Francesco and P Zinn-Justin *On the weighted enumeration of alternating sign matrices and descending plane partitions* arXiv: 1103.1176

Alternating Sign Matrices (ASMs)

$$\mathrm{ASM}(n) := \left\{ n \times n \text{ matrices} \;\middle|\; \begin{array}{l} \bullet \text{ each entry } 0,\, 1 \text{ or } -1 \\ \bullet \text{ at least one nonzero entry in each row \& column} \\ \bullet \text{ nonzero entries alternate in sign along each} \\ \quad \text{row \& column, starting \& ending with } 1 \end{array} \right\}$$

- Arose during study of Dodgson condensation algorithm for determinant evaluation *(Mills, Robbins, Rumsey 1982; Robbins, Rumsey 1986)*
- Many subsequent appearances in combinatorics, algebra, mathematical physics, ...

- e.g. ASM(3) =

$$\left\{ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix} \right\}$$

- e.g. $\begin{pmatrix}0&0&0&1&0&0\\0&1&0&-1&1&0\\1&-1&1&0&0&0\\0&0&0&1&0&0\\0&1&0&-1&0&1\\0&0&0&1&0&0\end{pmatrix} \in \mathrm{ASM}(6)$

Descending Plane Partitions (DPPs)

DPP(n) :=

$$\left\{ \begin{array}{l} \text{arrays } \begin{array}{cccccc} D_{11} & D_{12} & D_{13} & \ldots\ldots\ldots\ldots & \ldots & D_{1,\lambda_1} \\ & D_{22} & D_{23} & \ldots\ldots\ldots\ldots & D_{2,\lambda_2+1} & \\ & & D_{33} & \ldots\ldots & D_{3,\lambda_3+2} & \\ & & & \ddots \quad \therefore & & \\ & & & D_{tt} \ldots\ldots D_{t,\lambda_t+t-1} & & \end{array} \end{array} \middle| \begin{array}{l} \bullet \text{ each part (entry) a positive integer} \\ \bullet \text{ parts decrease weakly along rows} \\ \bullet \text{ parts decrease strictly down columns} \\ \bullet \ n \geq D_{11} > \lambda_1 \geq D_{22} > \ldots \geq D_{tt} > \lambda_t \end{array} \right\}$$

- Arose during study of cyclically symmetric plane partitions *(Andrews 1979)*

- e.g. $\text{DPP}(3) = \left\{ \emptyset, \begin{array}{cc} 3 & 3 \\ & 2 \end{array}, 2, 3\,3, 3, 3\,2, 3\,1 \right\}$

- e.g. $\begin{array}{ccccc} 6 & 6 & 6 & 5 & 2 \\ & 4 & 4 & 1 & \\ & & 3 & & \end{array} \in \text{DPP}(6)$

ASM Statistics

For $A \in \mathrm{ASM}(n)$

- $\nu(A) := \sum_{\substack{1 \le i < i' \le n \\ 1 \le j' \le j \le n}} A_{ij}\, A_{i'j'}$

  $= \#$ of 'inversions' in A

- $\mu(A) := \#$ of -1's in A

- $\rho(A) := \#$ of 0's to left of 1 in first row of A

- e.g. $A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

  $\nu(A) = 5, \quad \mu(A) = 3, \quad \rho(A) = 3$

DPP Statistics

For $D \in \text{DPP}(n)$

- $\nu(D) :=$ # of parts of D for which $D_{ij} > j - i$
  $=$ # of 'nonspecial' parts in D
- $\mu(D) :=$ # of parts of D for which $D_{ij} \leq j - i$
  $=$ # of 'special' parts in D
- $\rho(D) :=$ # of n's in (first row of) D

- e.g. $D = \begin{array}{ccccc} 6 & 6 & 6 & 5 & 2 \\ & 4 & 4 & 1 & \\ & & 3 & & \end{array} \in \text{DPP}(6)$

  (special parts: 2 & 1)

  $\nu(D) = 7,\ \ \mu(D) = 2,\ \ \rho(D) = 3$

ASM & DPP Generating Functions

- $Z_{\mathsf{ASM}}(n,x,y,z) := \sum_{A\in\mathsf{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z^{\rho(A)}$
- $Z_{\mathsf{DPP}}(n,x,y,z) := \sum_{D\in\mathsf{DPP}(n)} x^{\nu(D)}\, y^{\mu(D)}\, z^{\rho(D)}$

- e.g. ASM(3) =

$$\left\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix},\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix},\begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}\right\}$$

$$\mathsf{DPP}(3) = \left\{\emptyset,\ \begin{matrix}3&3\\&2\end{matrix},\ 2,\ 3\,3,\ 3,\ 3\,2,\ 3\,1\right\}$$

gives $Z_{\mathsf{ASM}}(3,x,y,z) = Z_{\mathsf{DPP}}(3,x,y,z) = 1 + x^3z^2 + x + x^2z^2 + xz + x^2z + xyz$

Main Result

Theorem $\left|\{A \in \mathsf{ASM}(n) \mid \nu(A) = p,\ \mu(A) = m,\ \rho(A) = k\}\right| =$
$\left|\{D \in \mathsf{DPP}(n) \mid \nu(D) = p,\ \mu(D) = m,\ \rho(D) = k\}\right|$ *for any* n, p, m, k

Equivalently $Z_{\mathsf{ASM}}(n,x,y,z) = Z_{\mathsf{DPP}}(n,x,y,z)$ *for any* n, x, y, z

- Conjectured *Mills, Robbins, Rumsey 1983*
- Proved *RB, Di Francesco, Zinn-Justin 2011*

Structure of proof

1. (a) Apply bijection between ASM(n) & {configurations of six-vertex model with domain-wall boundary conditions on $n \times n$ grid}
   (b) Use Izergin–Korepin formula & certain transformations to give $Z_{\mathsf{ASM}}(n,x,y,z) = \det M_{\mathsf{ASM}}(n,x,y,z)$ for $n \times n$ matrix $M_{\mathsf{ASM}}(n,x,y,z)$
2. (a) Apply bijection between DPP(n) & {certain nonintersecting lattice paths on $n \times n$ grid}
   (b) Use Lindström–Gessel–Viennot theorem to give $Z_{\mathsf{DPP}}(n,x,y,z) = \det M_{\mathsf{DPP}}(n,x,y,z)$ for $n \times n$ matrix $M_{\mathsf{DPP}}(n,x,y,z)$
3. Use elementary transformations of generating functions for entries of matrices to give $\det M_{\mathsf{ASM}}(n,x,y,z) = \det M_{\mathsf{DPP}}(n,x,y,z)$

Previously-Proved Special Cases

- $x=y=z=1$ (straight enumeration of ASMs, DPPs)

  $$\big|\mathsf{ASM}(n)\big| = \big|\mathsf{DPP}(n)\big| \;\left(= \prod_{i=0}^{n-1}\frac{(3i+1)!}{(n+i)!} = 1,2,7,42,429,7436,\ldots\right)$$

  ASM formula: conjectured *Mills, Robbins, Rumsey 1982*
  proved *Zeilberger 1996*, *Kuperberg 1996*
  DPP formula: *Andrews 1979*

- $x=y=1$ ('refined' enumeration of ASMs, DPPs)

  $$\big|\{A\in\mathsf{ASM}(n)\mid\rho(A)=k\}\big| = \big|\{D\in\mathsf{DPP}(n)\mid\rho(A)=k\}\big| \text{ (= product formula)}$$

  ASM formula: conjectured *Mills, Robbins, Rumsey 1983*, proved *Zeilberger 1996*
  DPP formula: *Mills, Robbins, Rumsey 1982*

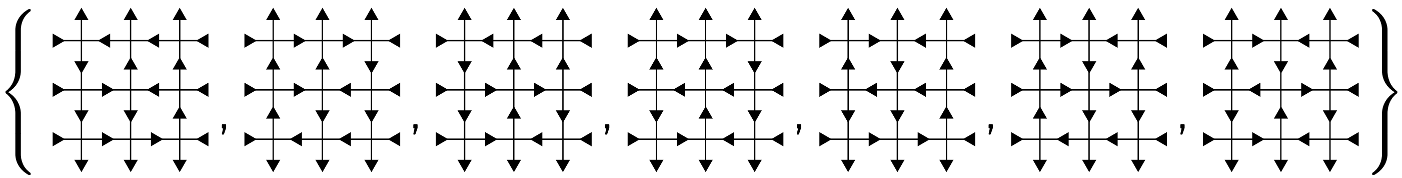
- $m=0$ (permutation matrices, DPPs with no special parts)
  Mills, Robbins, Rumsey 1983

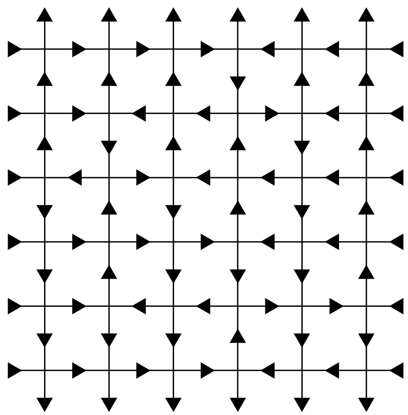
- $m=1$ (ASMs with single -1, DPPs with single special part)
  Lalonde 2002

- Certain other cases: see *RB, Di Francesco, Zinn-Justin 2011*

Configurations of Six-Vertex Model with Domain-Wall Boundary Conditions (DWBC)

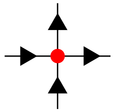
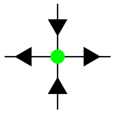
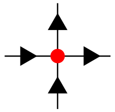
$$\mathrm{6VDW}(n) := \left\{ \begin{array}{l} \text{edge orientations} \\ \text{of } n \times n \text{ grid} \end{array} \middle| \begin{array}{l} \bullet \text{ 2 incoming \& 2 outgoing arrows at each} \\ \quad \text{internal vertex } (\Rightarrow \text{ 6 possible vertex conf'ns}) \\ \bullet \text{ upper \& lower boundary arrows all outgoing,} \\ \quad \text{left \& right boundary arrows all incoming} \end{array} \right\}$$

- e.g. $\mathrm{6VDW}(3) =$ 

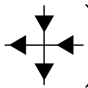
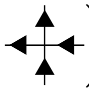
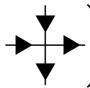
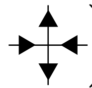
- e.g.  $\in \mathrm{6VDW}(6)$

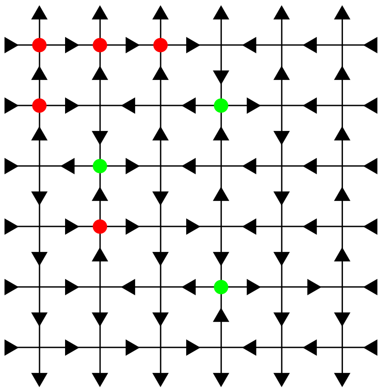
Six-Vertex Model with DWBC Statistics

For $C \in 6\text{VDW}(n)$

- $\nu(C) :=$ # of  vertex configurations in C
- $\mu(C) :=$ # of  vertex configurations in C
- $\rho(C) :=$ # of  vertex configurations in first row of C

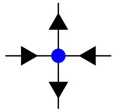
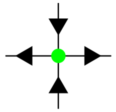
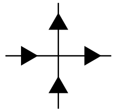
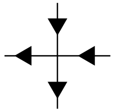
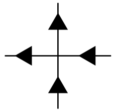
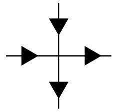
- numbers of other 4 vertex configurations in C satisfy

(# ) $= \nu(C)$, (# ) = (# ) $= \frac{n(n-1)}{2} - \nu(C) - \mu(C)$, (# ) $= \mu(C) + n$

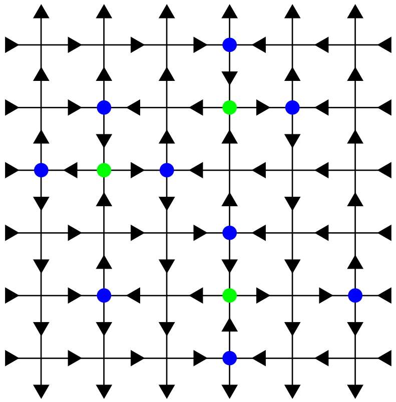
- e.g. 

$\nu(C) = 5, \quad \mu(C) = 3, \quad \rho(C) = 3$

ASM(n) − 6VDW(n) **Bijection**

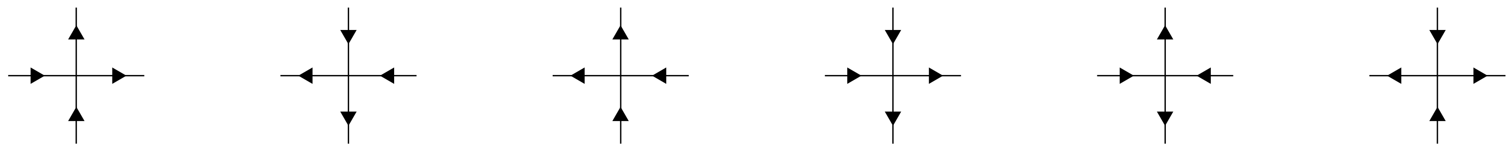
ASM		six-vertex model with DWBC
1	$\longleftrightarrow$	
-1	$\longleftrightarrow$	
0	$\longleftrightarrow$	 ,  ,  or 

- Gives bijection between $\{A \in \text{ASM}(n) \mid \nu(A) = p,\ \mu(A) = m,\ \rho(A) = k\}$ & $\{C \in \text{6VDW}(n) \mid \nu(C) = p,\ \mu(C) = m,\ \rho(C) = k\}$
  (Elkies, Kuperberg, Larsen, Propp 1992)

- e.g. $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \longleftrightarrow$ 

Izergin–Korepin Formula

- Integrable vertex weights:



$\bar{a}(u,v)$ $\quad$ $\bar{a}(u,v)$ $\quad$ $\bar{b}(u,v)$ $\quad$ $\bar{b}(u,v)$ $\quad$ $\bar{c}(u,v)$ $\quad$ $\bar{c}(u,v)$

$$\bar{a}(u,v) := uq - \tfrac{1}{vq} \qquad \bar{b}(u,v) := \tfrac{u}{q} - \tfrac{q}{v} \qquad \bar{c}(u,v) := (q^2 - \tfrac{1}{q^2})\sqrt{\tfrac{u}{v}}$$

- $\dfrac{\bar{a}(u,v)^2 + \bar{b}(u,v)^2 - \bar{c}(u,v)^2}{\bar{a}(u,v)\,\bar{b}(u,v)} = q^2 + q^{-2} \implies$ Yang-Baxter equation satisfied

- Izergin–Korepin formula for partition function of six-vertex model with DWBC:

$$Z_{\text{6VDW}}(u_1,\ldots,u_n;v_1,\ldots,v_n) := \sum_{C\in \text{6VDW}(n)} \prod_{i,j=1}^{n} \begin{pmatrix}\text{weight at vertex } (i,j) \text{ with}\\ \text{parameters } u_i,\, v_j \text{ in config'n } C\end{pmatrix}$$

$$= \frac{\prod_{i=1}^{n} \bar{c}(u_i,v_i)\ \prod_{i,j=1}^{n} \bar{a}(u_i,v_j)\,\bar{b}(u_i,v_j)}{\prod_{1\le i<j\le n}(u_i-u_j)(v_j^{-1}-v_i^{-1})} \det_{1\le i,j\le n}\left(\frac{1}{\bar{a}(u_i,v_j)\,\bar{b}(u_i,v_j)}\right) \qquad \textit{(Izergin 1987)}$$

$$= \frac{\prod_{i=1}^{n} u_i^{1/2} v_i^{n+1/2}\ \prod_{i,j=1}^{n} \bar{a}(u_i,v_j)\,\bar{b}(u_i,v_j)}{\prod_{1\le i<j\le n}(u_i-u_j)(v_i-v_j)} \det_{1\le i,j\le n}\left(\frac{1}{u_iv_j-q^2} - \frac{1}{u_iv_j-q^{-2}}\right)$$

ASM Determinant

Lemma $Z_{\mathrm{ASM}}(n,x,y,1) = \det_{0\le i,j\le n-1} M_{\mathrm{ASM}}(n,x,y)_{ij}$

where $M_{\mathrm{ASM}}(n,x,y)_{ij} := (1-\omega)\,\delta_{ij} + \omega \sum_{k=0}^{\min(i,j)} \binom{i}{k}\binom{j}{k} x^k\, y^{i-k}$

& ω *is either solution of* $y\,\omega^2 + (1-x-y)\omega + x = 0$

- Similar result for $z \neq 1$ with last column of $M_{\mathrm{ASM}}(n,x,y)$ modified
- For straight enumeration $M_{\mathrm{ASM}}(n,1,1)_{ij} = e^{\pm i\pi/3}\,\delta_{ij} + e^{\mp i\pi/3}\binom{i+j}{i}$

Proof

Let $a := \bar{a}(r,r) = qr - \frac{1}{qr}$, $b := \bar{b}(r,r) = \frac{r}{q} - \frac{q}{r}$, $c := \bar{c}(r,r) = q^2 - \frac{1}{q^2}$, $x := (\frac{a}{b})^2$, $y := (\frac{c}{b})^2$

Then $Z_{\mathrm{ASM}}(n,x,y,1) = \sum_{A\in\mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)} = \sum_{A\in\mathrm{ASM}(n)} (\frac{a}{b})^{2\,\nu(A)}\,(\frac{c}{b})^{2\,\mu(A)}$

$= \sum_{C\in 6\mathrm{VDW}(n)} (\frac{a}{b})^{2\,\nu(C)}\,(\frac{c}{b})^{2\,\mu(C)}$ [using ASM(n) – 6VDW(n) bijection]

$= \frac{1}{b^{n(n-1)}\,c^n} \sum_{C\in 6\mathrm{VDW}(n)} a^{2\nu(C)}\, b^{n(n-1)-2\nu(C)-2\mu(C)}\, c^{2\mu(C)+n}$

$= \frac{1}{b^{n(n-1)}\,c^n}\, Z_{6\mathrm{VDW}}(r,\ldots,r;r,\ldots,r)$ [using properties of vertex config'n numbers]

$$= \frac{r^{n(n+1)}\,a^{n^2}b^n}{c^n} \lim_{\substack{u_1,\dots,u_n\to r\\ v_1,\dots,v_n\to r}} \frac{1}{\prod_{1\le i<j\le n}(u_i-u_j)(v_i-v_j)} \det_{1\le i,j\le n}\left(\frac{1}{u_iv_j-q^2}-\frac{1}{u_iv_j-q^{-2}}\right)$$

[using Izergin–Korepin formula]

$$= \frac{r^{n(n+1)}\,a^{n^2}b^n}{c^n} \det_{0\le i,j\le n-1}\left([u^iv^j]\left(\frac{1}{(u+r)(v+r)-q^2}-\frac{1}{(u+r)(v+r)-q^{-2}}\right)\right)$$

[transforming det using divided differences, where $[u^iv^j]$ denotes coeff. of u^iv^j in series expansion]

$$= \frac{r^{n(n+1)}\,a^{n^2}b^n}{c^n} \det\left(\frac{1}{q^{-2}-r^2}\, L\big(\tfrac{r}{q^{-2}-r^2},\tfrac{1}{qr}\big)\, L\big(\tfrac{r}{q^{-2}-r^2},\tfrac{1}{qr}\big)^t - \frac{1}{q^2-r^2}\, L\big(\tfrac{r}{q^2-r^2},\tfrac{q}{r}\big)\, L\big(\tfrac{r}{q^2-r^2},\tfrac{q}{r}\big)^t\right)$$

[defining lower triangular matrix $L(\alpha,\beta)_{ij} := \binom{i}{j}\,\alpha^i\,\beta^j,\ \ 0\le i,j\le n-1$]

$$= \frac{r^{n(n+1)}\,a^{n^2}b^n}{c^n\,(q^{-1}-qr^2)^{n(n-1)}} \det\left(\frac{1}{q^{-2}-r^2}\,I - \frac{1}{q^2-r^2}\, L\big(\tfrac{r}{q^{-2}-r^2},\tfrac{1}{qr}\big)^{-1} L\big(\tfrac{r}{q^2-r^2},\tfrac{q}{r}\big)\, L\big(\tfrac{r}{q^2-r^2},\tfrac{q}{r}\big)^t \left(L\big(\tfrac{r}{q^{-2}-r^2},\tfrac{1}{qr}\big)^t\right)^{-1}\right)$$

$$= \frac{r^{n(n+1)}\,a^{n^2}b^n}{c^n\,(q^{-1}-qr^2)^{n(n-1)}} \det\left(\frac{1}{q^{-2}-r^2}\,I + \frac{1}{r^2-q^2}\, L\big(\tfrac{q^2-q^{-2}}{q^{-1}r-qr^{-1}},\tfrac{qr-(qr)^{-1}}{q^2-q^{-2}}\big)\, L\big(\tfrac{q^2-q^{-2}}{q^{-1}r-qr^{-1}},\tfrac{qr-(qr)^{-1}}{q^2-q^{-2}}\big)^t\right)$$

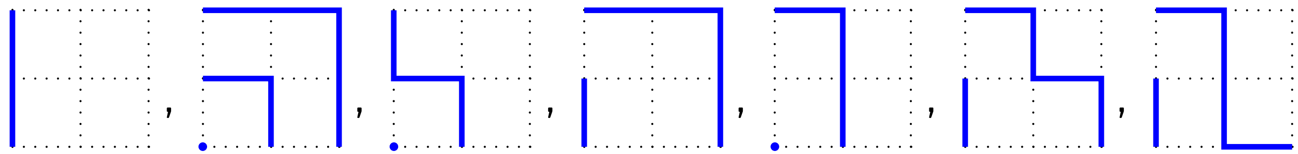
[using binomial coefficient properties]

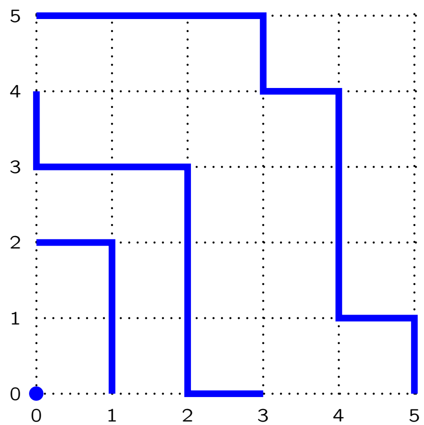
$$= \det\Big((1-\omega)\,I + \omega\, L\big(\tfrac{c}{b},\tfrac{a}{c}\big)\, L\big(\tfrac{c}{b},\tfrac{a}{c}\big)^t\Big) \qquad \text{[defining } \omega := \tfrac{r^2-q^{-2}}{q^2-q^{-2}}\text{]}$$

$$= \det M_{\mathsf{ASM}}(n,x,y), \ \ \text{with } y\,\omega^2+(1-x-y)\omega+x=0 \qquad \square$$

Nonintersecting Lattice Paths

$$\mathrm{NILP}(n) := \left\{ \begin{array}{l|l} \text{nonintersecting path} & P \text{ consists of paths from } (0, \lambda_{i-1} - 1) \text{ to } (\lambda_i, 0) \\ \text{sets } P \text{ on } n \times n \text{ grid} & \text{for each } i = 1, \ldots, t+1, \text{ with each step} \\ & \text{rightward or downward, for some } 0 \leq t \leq n-1 \\ & \text{\& } n = \lambda_0 > \lambda_1 > \ldots > \lambda_t > \lambda_{t+1} = 0 \end{array} \right\}$$

- e.g. NILP(3) = {  }

- e.g. 

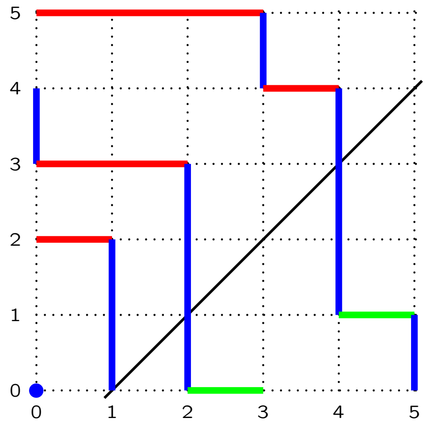
$\in$ NILP(6) $\quad [(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (6, 5, 3, 1, 0)]$

Nonintersecting Lattice Path Statistics

For $P \in \mathrm{NILP}(n)$

- $\nu(P) :=$ # of rightward steps in P above line $\{(i, i-1)\}$
- $\mu(P) :=$ # of rightward steps in P below line $\{(i, i-1)\}$
- $\rho(P) :=$ # of rightward steps in P in top row of grid

- e.g.



$\nu(P) = 7, \quad \mu(P) = 2, \quad \rho(P) = 3$

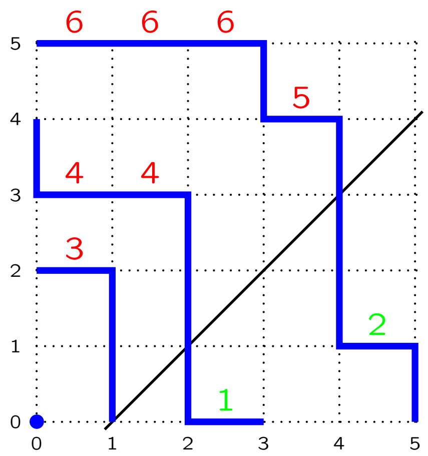
DPP(n) − NILP(n) **Bijection**

DPP		nonintersecting path set
$D_{ij} - 1$	$\longleftrightarrow$	height of $(j - i + 1)$th rightward step of ith path from top

- Gives bijection between $\{D \in \text{DPP}(n) \mid \nu(D) = p,\ \mu(D) = m,\ \rho(D) = k\}$ & $\{P \in \text{NILP}(n) \mid \nu(P) = p,\ \mu(P) = m,\ \rho(P) = k\}$ *(Lalonde 2002)*

- e.g.

$$\begin{array}{ccccc} 6 & 6 & 6 & 5 & 2 \\ & 4 & 4 & 1 & \\ & & 3 & & \end{array} \quad \longleftrightarrow$$



Lindström–Gessel–Viennot Theorem

- Assign weight $w(e)$ to each edge e of lattice
- Let $W(p) := \prod_{\text{edges } e \text{ of } p} w(e)$ for any lattice path p
- Let $\mathcal{P}(j,i) :=$ {lattice paths from $(0,j)$ to $(i,0)$ with each step rightward or downward}
- Let $\mathcal{N}(j_1,\dots,j_n;i_1,\dots,i_n) :=$

$$\left\{ \text{path sets } P \,\middle|\, \begin{array}{l} \bullet\ P \text{ consists of path of } \mathcal{P}(j_k,i_k) \text{ for each } k=1,\dots,n \\ \bullet \text{ different paths of } P \text{ do not intersect} \end{array} \right\}$$

Then

$$\boxed{\sum_{P\in\mathcal{N}(j_1,\dots,j_n;i_1,\dots,i_n)} \prod_{p\in P} W(p) = \det_{1\le i,j\le n}\Big(\sum_{p\in\mathcal{P}(j,i)} W(p)\Big)}$$

(Lindström 1973; Gessel, Viennot 1989)

DPP Determinant

Lemma $Z_{\text{DPP}}(n,x,y,1) = \det_{0\le i,j\le n-1} M_{\text{DPP}}(n,x,y)_{ij}$

where $M_{\text{DPP}}(n,x,y)_{ij} := -\delta_{i,j+1} + \sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} x^k\, y^{i-k}$

- Similar result for $z \neq 1$ with last column of $M_{\text{DPP}}(n,x,y)$ modified
- For straight enumeration $M_{\text{DPP}}(n,1,1)_{ij} = -\delta_{i,j+1} + \binom{i+j}{i}$

Proof

For edge e, let $w(e) = \begin{cases} x, & e \text{ horizontal \& above line } \{(i,i-1)\} \\ y, & e \text{ horizontal \& below line } \{(i,i-1)\} \\ 1, & e \text{ vertical} \end{cases}$

Then DPP(n) – NILP(n) bijection,

fact that $\text{NILP}(n) = \bigcup_{n-1\ge\lambda_1>\ldots>\lambda_t\ge 1} \mathcal{N}(n-1,\lambda_1-1,\ldots,\lambda_t-1;\lambda_1,\ldots,\lambda_t,0)$

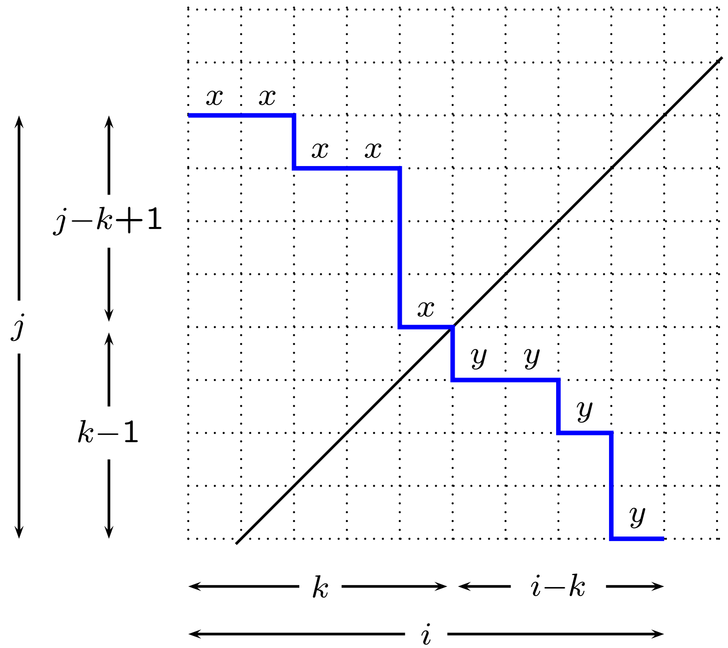
& Lindström–Gessel–Viennot theorem give

$$Z_{\text{DPP}}(n,x,y,1) = \sum_{n-1\ge\lambda_1>\ldots>\lambda_t\ge 1} \det_{\substack{i=0,\lambda_t,\ldots,\lambda_1 \\ j=\lambda_t-1,\ldots,\lambda_1-1,n-1}} \left(\sum_{p\in\mathcal{P}(j,i)} W(p) \right)$$

Identity $\det_{0\le i,j\le n-1}(-\delta_{i,j+1}+A_{ij}) = \sum_{T\subset\{1,\dots,n-1\}} \det\begin{pmatrix}\text{submatrix of } A \text{ with rows \& columns ind-}\\ \text{exed by } \{0\}\cup T \ \&\ \{t-1 \mid t\in T\}\cup\{n-1\}\end{pmatrix}$

gives $Z_{\mathrm{DPP}}(n,x,y,1) = \det_{0\le i,j\le n-1}\Bigl(-\delta_{i,j+1} + \sum_{p\in\mathcal{P}(j,i)} W(p)\Bigr)$

Geometry gives $\displaystyle\sum_{p\in\mathcal{P}(j,i)} W(p) = \sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} x^k\, y^{i-k}$



Therefore $Z_{\mathrm{DPP}}(n,x,y,1) = \det M_{\mathrm{DPP}}(n,x,y)$ □

Equality of ASM & DPP Determinants

Lemma $\det M_{\mathsf{ASM}}(n,x,y) = \det M_{\mathsf{DPP}}(n,x,y)$

where (as before) $M_{\mathsf{ASM}}(n,x,y)_{ij} = (1-\omega)\,\delta_{ij} \;+\; \omega \sum_{k=0}^{\min(i,j)} \binom{i}{k}\binom{j}{k}\, x^k\, y^{i-k}$

$$M_{\mathsf{DPP}}(n,x,y)_{ij} = -\delta_{i,j+1} + \sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} x^k\, y^{i-k}, \quad 0 \le i,j \le n-1$$

& ω *satisfies* $y\,\omega^2 + (1-x-y)\omega + x = 0$

<u>Proof</u>

Let $$g_{\mathsf{ASM}}(x,y;u,v) := \frac{1-\omega}{1-uv} + \frac{\omega}{1-yu-v-(x-y)uv}$$

$$g_{\mathsf{DPP}}(x,y;u,v) := \frac{1-u}{(1-v)(1-uv)} + \frac{xu}{(1-v)(1-yu-v-(x-y)uv)}$$

Then $M_{\mathsf{ASM}}(n,x,y)_{ij} = [u^i v^j]\, g_{\mathsf{ASM}}(x,y;u,v)$ & $M_{\mathsf{DPP}}(n,x,y)_{ij} = [u^i v^j]\, g_{\mathsf{DPP}}(x,y;u,v)$

where $[u^i v^j]$ denotes coefficient of $u^i v^j$ in series expansion

Eq'n satisfied by ω implies $(1+(x-\omega y-1)u)\, g_{\mathsf{ASM}}(x,y;u,v) = (1+(\omega-1)v)\, g_{\mathsf{DPP}}(x,y;u,v)$

Therefore $(I+(x-\omega y-1)S)\, M_{\mathsf{ASM}}(n,x,y) = M_{\mathsf{DPP}}(n,x,y)\,(I+(\omega-1)S^t)$ with $S_{ij} := \delta_{i,j+1}$

Therefore $\det M_{\mathsf{ASM}}(n,x,y) = \det M_{\mathsf{DPP}}(n,x,y)$ $\square$

Summary

- ASM(n) – 6VDW(n) bijection & transformed Izergin–Korepin formula give $Z_{\text{ASM}}(n,x,y,1) = \det M_{\text{ASM}}(n,x,y)$
- DPP(n) – NILP(n) bijection & Lindström–Gessel–Viennot theorem give $Z_{\text{DPP}}(n,x,y,1) = \det M_{\text{DPP}}(n,x,y)$
- Elementary transformations give $\det M_{\text{ASM}}(n,x,y) = \det M_{\text{DPP}}(n,x,y)$
- Therefore $Z_{\text{ASM}}(n,x,y,1) = Z_{\text{DPP}}(n,x,y,1)$
- Similar process with last columns of $M_{\text{ASM}}(n,x,y)$ & $M_{\text{DPP}}(n,x,y)$ modified gives $Z_{\text{ASM}}(n,x,y,z) = Z_{\text{DPP}}(n,x,y,z)$
- Therefore ASM – DPP conjecture valid
- Byproduct of proof: new determinant formulae for weighted ASM enumeration, or for partition function of homogeneous six-vertex model with DWBC e.g. $c^n \det_{0\le i,j\le n-1}\left(-b^{2i}\,\delta_{i,j+1} + \sum_{k=0}^{\min(i,j+1)} \binom{i-1}{i-k}\binom{j+1}{k} a^{2k}\, c^{2(i-k)}\right)$

Work in Progress

- Doubly-refined theorem involving further boundary statistic

- Theorem involving vertically symmetric ASMs & DPPs invariant under certain Mills–Robbins–Rumsey symmetry operation

Further Questions

- ASM properties corresponding to certain natural DPP properties
  e.g. ASM statistic corresponding to # of rows of DPP?
  ASM statistic corresponding to sum of parts of DPP?

- DPP properties corresponding to certain natural ASM properties
  e.g. DPP statistics corresponding to positions of 1 in first or last column of ASM?
  DPP operations corresponding to $\pi/2$-rotation or transposition of ASM?

- Natural bijection between ASMs & DPPs?