Cardiff Lecture notes

0) Philosophy

a) Geometry $\Rightarrow$ Algebra:

Algebra is about rings and algebras, and modules over them.

Take an assoc. ring A.

Question: How to define A?

Answer 1: Generators and relations:

$A = \text{Free}(x_1, \dots, x_n) / \langle r_1, \dots, r_m \rangle$ (*)

$r_1, \dots, r_m$ - generators in the 2-sided ideal of relations.

Rem: This is not good:

- A can be presented in the form (*) in many ways
- Bad for description of morphisms $\varphi: A_1 \to A_2$
- (*) does not tell much about properties of the algebra A.

Answer 2 : Suppose that A is commutative, over the field $\mathbb{R}$ (or $\mathbb{C}$) then a nice way to talk about A:

$A = \mathrm{Fun}(X)$

- Here X is a "space" (depending on the language, X - top. space, $\mathbb{R}$-mfld, $\mathbb{C}$-mfld, algebraic variety$/\mathbb{C}$)
- $\mathrm{Fun}(X)$ is the algebra of functions on X which, depending on the setting, can be
  - Continuous
  - C^∞
  - holomorphic
  - polynomial

$\varphi: A_1 \longrightarrow A_2$ comes from

$f: Y \longrightarrow X$, $A_1 = \mathrm{Fun}(X)$, $A_2 = \mathrm{Fun}(Y)$

$\varphi: \alpha \longmapsto \alpha \circ f$

Rem: The idea culminates in the notion of Spectrum of a ring and leads to algebraic geometry

Our goal is to do something similar for non-commutative algebras.

b) <u>Categorification</u> :

Let $\mathcal{A}$ be an abelian (or triangulated) category.

<u>Def</u>: The Grothendieck group $K(\mathcal{A})$ is a free Abelian group $\langle [X] \mid X$ - class of iso of objects in $\mathcal{A} \rangle_{\mathbb{Z}}$ quotient by the relations:

. For $0 \to X \to Y \to Z \to 0$ - short exact sequence in $\mathcal{A}$,

$[Y] = [X] + [Z]$.

<u>Rem</u>: A similar definition works for a triangulated category $\mathcal{D}$.

Let $\mathcal{A}$ be monoidal, with exact tensor product (resp. $\mathcal{D}$ - monoidal, with exact tensor product)

$\otimes \; \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$

Put $[X] \cdot [Y] = [X \otimes Y]$

Get an assoc. ring structure on $K(\mathcal{A})$.

This is called the <u>decategorification</u> of $\mathcal{A}$: $\mathcal{A} \rightsquigarrow K(\mathcal{A})$

(resp. $\mathcal{D} \rightsquigarrow K(\mathcal{D})$)

<u>Rem</u>: Categorification of a given algebra A is search for a category $\mathcal{A}$ (resp $\mathcal{D}$) such that $A = K(\mathcal{A})$ (or $A = K(\mathcal{D})$)

<u>Advantage of categorification</u>:

In good cases, $\mathcal{A}$ comes with distinguished classes of objects, e.g.:

- simple obj. $\{S_\alpha\}$
- indecomposable projective obj $\{P_\beta\}$
- indecomposable injective obj $\{I_\gamma\}$

...

$\{[S_\alpha]\}$, $\{[P_\beta]\}$, $\{[I_\gamma]\}$ form distinguished bases in $K(\mathcal{A}) \otimes \mathbb{Q}$.

These bases are related by transformation matrices, e.g. $M^{\{[L_\alpha]\}}_{\{[P_\beta]\}}$. The entries of the matrices tell a lot about $\mathcal{A}$.

In the other direction, $A = K(\mathcal{A}) \otimes \mathbb{Q}$, then knowing properties of $\mathcal{A}$ $\Longrightarrow$ deep study of the algebra A

1) Toy example of a Hecke algebra

Consider a finite group K with a subgroup H

a) First $H = \{1\}$

Def. $\mathcal{H} = \mathcal{H}(K)$ as a $\mathbb{Q}$-vector space equals $\mathrm{Fun}(K, \mathbb{Q})$, with the product

$$* : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$$

$$(f_1 * f_2)(k) = m_*(f_1 \times f_2)(k)$$

$$= \frac{1}{\# K} \sum_{k_1 k_2 = k} f_1(k_1) f_2(k_2),$$

here

$$k_1, k_2, k \in K$$

$$f_1 \times f_2 : K \times K \longrightarrow \mathbb{Q}$$

$$m : K \times K \longrightarrow K$$

(product in the group)

Claim: (i) $(\mathcal{H}, *)$ is an associative algebra / $\mathbb{Q}$

(ii) $\mathcal{H} \cong \mathbb{Q}[K]$, the group algebra of K

b) Reformulation useful for generalizations:

Rem.: K acts on $K \times K$ diagonally:

$$k, (k_1, k_2) \longmapsto (k_1 k^{-1}, k k_2)$$

We have $\mathrm{Fun}(K) = \mathrm{Fun}^K(K \times K)$ - the $\mathbb{Q}$-vector space of K-invariant functions on $K \times K$

Claim: as a vector space, $\mathcal{H} = \mathrm{Fun}^K(K \times K)$, and the product in these terms is described as follows:

(★★)

$$\begin{tikzcd} & K \times K \times K \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ K \times K & K \times K & K \times K \end{tikzcd}$$

$$f_1 * f_2 = \pi_{13*}\left(\pi_{12}^*(f_1) \cdot \pi_{23}^*(f_2)\right)$$

Here $\pi_{12}^* : f \longmapsto f \circ \pi_{12}$

π_{13*} - summation over the fibers

Rem: In many other settings, the formulas similar to (**) define associative products.

b) $K \supset H$ - a finite group with a subgroup

Def: $\mathcal{H}(K,H) := \mathrm{Fun}^K(K/H \times K/H)$ as a $\mathbb{Q}$-vector space, the algebra structure is given by

$$\begin{tikzcd}
 & K/H \times K/H \times K/H \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\
K/H \times K/H & K/H \times K/H & K/H \times K/H
\end{tikzcd}$$

$$f_1 * f_2 = \pi_{13*}\left(\pi_{12}^*(f_1) \cdot \pi_{23}^*(f_2)\right)$$

is called the (finite) Iwahori-Hecke algebra of the pair $K \supset H$.

c) <u>Classical example</u>

$k = \mathbb{F}_q$ - a finite field.

Take $K = SL(n, \mathbb{F}_q)$,

H = upper triangular matrices M, $\det(M) = 1$.

Introduce the <u>Flag variety</u> (a finite set):

$$Fl_q^n = \left\{ V_\cdot \text{ - complete flags in } \mathbb{F}_q^n \;\middle|\; \begin{array}{l} V_\cdot = \{0 \subset V_1 \subset \dots \subset V_n = \mathbb{F}_q^n\} \\ \dim V_i = i \end{array} \right\}$$

$V_\cdot^0 = k \subset k^2 \subset \dots \subset k^n$ - standard coordinate flag.

<u>Rem.</u> (i) K acts on Fl_q^n since it acts on $\mathbb{F}_q^n$

(ii) $\mathrm{Stab}_K(V_\cdot^0) = H$

(iii) $\mathrm{Orb}_K(V_\cdot^0) = Fl$

(iv) thus $Fl_n^q \cong K/H$ as a set.

Consider $\mathcal{H}_q = \mathcal{H}(K, H) =$

$= \mathrm{Fun}^{SL(n, \mathbb{F}_q)}(\# Fl_n^q \times \# Fl_n^q)$ -

Proposition: (i) K-orbits on $Fl_n^q \times Fl_n^q$ ~~are~~ enumerated by elements of the symmetric group $\sigma \in S_n$

(ii) $\mathcal{H}_q$ has a basis $\{T_\sigma, \sigma \in S_n\}$

(iii) $\mathcal{H}_q$ is generated by $\{T_i, i=1,\dots,n-1$

$T_i = T_{(i\,i+1)}\}$

(iv) the relations on $\{T_i\}$ are:

$$\cdot\ T_iT_j = T_jT_i, \quad |i-j|>1 \qquad (\star\star\star)$$

$$\cdot\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$$

$$\cdot\ (T_i - q)(T_i+1) = 0$$

Rem: (i) T_i come from delta-functions of K-orbits on $Fl_n^q \times Fl_n^q$,

(ii) For $\sigma = (i\,i+1)$ the corresponding orbit consists of

$\{(V_\cdot^1, V_\cdot^2)$, $V_j^1 = V_j^2$, $j \neq i$, $V_i^1 \neq V_i^2$

(iii) this allows to check the relations $(\star\star\star)$

Rem: (i) $\mathcal{H}_q$ in this presentation by generators and relations is defined over $\mathbb{Q}(q)$ where q is a formal variable.

(ii) T_i are invertible in $\mathcal{H}_q$

2) Kazhdan-Lusztig canonical basis for $\mathcal{H}_q$, integral properties and the need for geometric machinery

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From now on G - an algebraic group $/\mathbb{C}$, simple. $B \subset G$ - the Borel subgroup.

Rem. E.g. $G = SL(n, \mathbb{C})$, $B \subset G$ - upper triangular matr.

$Fl_G := G/B$. This is a projective algebraic variety $/\mathbb{C}$.

Fix a maximal torus $T \subset B \subset G$

Rem. E.g. for $G = SL(n, \mathbb{C})$
T = diagonal matr. with $\det(A) = 1$

Recall: the Weyl group
$$W = \mathrm{Norm}_G(T)/T$$

Rem. E.g. for $G = SL(n, \mathbb{C})$
$W = S_n$, the symmetric group.

Proposition G-orbits in $Fl_G \times Fl_G$ are in bijection with $w \in W$.

Topological black box:
We work in the derived category of mixed constructible sheaves on $Fl_G \times Fl_G$ equivariant wrt to the diagonal G-action
$$D^{b,G}_{const,mix}(Fl_G \times Fl_G) =: \mathcal{H}ecke(G, B)$$

Rem: We do not define the category, only state

Theorem: (i) $\mathcal{H}ecke(G,H)$ is a monoidal triangulated category

(ii) $K(\mathcal{H}ecke(G,H))$ is isomorphic to $\mathcal{H}_q$ (for the corresponding Weyl group) as a $\mathbb{Z}[q,q^{-1}]$-algebra.

Rem: The q-action comes from the additional grading on the category hidden in the word mixed

Kazhdan-Lusztig involution:

$- : \mathcal{H}_q \to \mathcal{H}_q$, $q \mapsto q^{-1}$,

$T_i \longmapsto T_i^{-1}$

Rem: For technical reasons, work over $\mathbb{Z}[q^{\pm 1/2}]$.

Th (Kazhdan–Lusztig)
∃! basis of $\mathcal{H}_q$ $\{C'_w, w \in W\}$
such that
a) $\overline{C'_w} = C'_w$
b) $C'_w = (q^{-1/2})^{\ell(w)} \sum_{y \le w} P_{y,w} T_y$

$P_{w,w} = 1$

. Here $P_{y,w} \in \mathbb{Q}(q^{\pm 1/2})$ – elements of the matr relating $\{C'_w\}$ and $\{T_y\}$
. $\le$ stands for the Bruhat order on $\{w \in W\}$

Rem: All this can be proved algebraically, but here is the result which was proved without use of algebraic geometry and topology only in 2013 (Elias, Williamson)

. $P_{yw} \in \mathbb{Z}[q]$
they are called Kazhdan–Lusztig polynomials.

Geometric interpretation:

Recall that

$$\mathcal{H}_q = K\left(\mathcal{D}^{b,G}_{const,\,mix}(Fl_G \times Fl_G)\right)$$

Theorem: (i) Verdier anti-equivalence of the category corresponds to Kazhdan-Lusztig involution

(ii) the classes of IC sheaves corresponding to G-orbits

$[IC_w] = C_w$ (canonical basis in $\mathcal{H}_q$)

3) Further developments and generalizations

a) Replace Fl_G by the affine flag variety $Fl_{G,af}$ and G - by the loop group LG.

Consider the <u>affine Hecke category</u>

$$\mathcal{H}ecke_{af} := D^{b,mix}_{const, LG}(Fl_{af} \times Fl_{af})$$

It is also a monoidal category and $K(\mathcal{H}ecke_{af})$ is the <u>affine Iwahori-Hecke algebra</u>

b) <u>Replace</u>: coherent sheaves instead of constructible sheaves

<u>Def</u>: Coherent Hecke category

$$QCHecke(G,B) := D^b Coh^G(Fl_G \times Fl_G)$$

The same formula

$$\mathcal{F}_1 * \mathcal{F}_2 := \pi_{13*}(\pi_{12}^*(\mathcal{F}_1) \otimes \pi_{23}^*(\mathcal{F}_2))$$

defines a monoidal structure on $QCHecke(G,B)$