Multicentric calculus: polynomial as a new variable

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Abstract

We outline some ideas on multicentric calculus.
The basic technical tool is to change variable z to w as follows:

$$w = p(z).$$

Here p is a polynomial in the complex plane (or real line) with simple roots and therefore the change is locally near zeros conformal. However, we make this change **globally**, and since this is then a **many-to-one** mapping, we compensate the "loss of information" by introducing a vector valued function f to represent the original scalar function φ. We call the approach as " multicentric calculus".

Since sets bounded by lemniscates $|p(z)| \leq \rho$ are mapped to discs and on the other hand Hilbert's lemniscates theorem says that any compact set K on the complex plane with connected complement can be approximated with lemniscates, the calculus allows to replace functions in such complicated sets by functions defined on discs $|w| \leq \rho$.

To recover the functions in the original variable one just takes a linear combination

$$\varphi(z) = \sum_{j=1}^{d} \delta_j(z) f_j(w) \text{ where } w = p(z).$$

Here δ_j is the "Lagrange" interpolation polynomial, taking value 1 at one root of p and vanishing in the others.

Multicentric calculus can be used in an effective way to compute $\varphi(A)$ when $p(A)$ is much "nicer" than A, in particular, when an effective functional calculus exists for $p(A)$ but is not directly available for A.

We discuss two directions, one based on holomorphic functional calculus (and references [3-5] cover much of that). The other direction then generalizes the simple calculus where continuous functions are evaluated at diagonalizable matrices as follows:

$$\varphi(A) = T\varphi(D)T^{-1}.$$

In particular, one need **not** to assume that φ is differentiable if the matrix A has nontrivial Jordan blocks. Such functions φ form a natural Banach algebra. This is so far unpublished.

References

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