Variance Covariance Matrices for Linear Regression with Errors in both Variables

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Chapter 1 Introduction

In the technical report by Gillard and Iles [9], several method of moments based estimators for the errors in variables model were introduced and discussed. Broadly speaking, the method of moments estimators can be divided into two separate classifications. These are:

- Restricted parameter space estimators
- Higher moment based estimators

Restricted parameter space estimators These estimators are based on first and second order moments, and are derived by assuming that a parameter (or a function of parameters) is known a priori.

Higher moment based estimators If the data displays sufficient skewness and kurtosis, one may estimate the parameters of the model by appealing to estimators based on third and fourth order moments. Although these estimators have the initial appeal of avoiding a restriction on the parameter space, they must be used with caution.

Details of how these estimators were derived, and how they may be used in practise were again given by Gillard and Iles [9]. This present technical report aims to provide the practitioner with further details concerning asymptotic variance and covariances for both the restricted cases, and higher moment based estimators. In this way, this report can be viewed as a direct sequel to the technical report by Gillard and Iles [9].

Chapter 2

The Simple Linear Errors in Variables Model

Consider two variables, ξ and η which are linearly related in the form

$$\eta_i = \alpha + \beta \xi_i, \quad i = 1, \dots, n$$

However, instead of observing ξ and η , we observe

$$\begin{aligned} x_i &= \xi_i + \delta_i \\ y_i &= \eta_i + \varepsilon_i = \alpha + \beta \xi_i + \varepsilon_i \end{aligned}$$

where δ and ε are considered to be random error components, or noise.

It is assumed that $E[\delta_i] = E[\varepsilon_i] = 0$ and that $Var[\delta_i] = \sigma_{\delta}^2, Var[\varepsilon_i] = \sigma_{\varepsilon}^2$ for all i. Also the errors δ and ε are mutually uncorrelated. Thus

$$Cov[\delta_i, \delta_j] = Cov[\varepsilon_i, \varepsilon_j] = 0, i \neq j$$

$$Cov[\delta_i, \varepsilon_j] = 0, \forall i, j$$
(2.1)

There have been several reviews of errors in variables methods, notably Casella and Berger [2], Cheng and Van Ness [3], Fuller [7], Kendall and Stuart [14] and Sprent [15]. Unfortunately the notation has not been standardised. This report closely follows the notation set out by Cheng and Van Ness [3] but for convenience, it has been necessary to modify parts of their notation. Errors in variables modelling can be split into two general classifications defined by Kendall [12], [13], as the functional and structural models. The fundamental difference between these models lies in the treatment of the $\xi'_i s$

The functional model This assumes the $\xi'_i s$ to be unknown, but fixed constants μ_i .

The structural model This model assumes the $\xi'_i s$ to be a random sample from a random variable with mean μ and variance σ^2 .

It is the linear structural model that is the main focus of this technical report.

2.1 The Method of Moments Estimating Equations

The method of moments estimating equations follow from equating population moments to their sample equivalents. By using the properties of ξ , δ and ε detailed above, the population moments can be written in terms of parameters of the model. This was also done by Kendall and Stuart [14], amongst others.

$$\begin{split} E[X] &= E[\xi] = \mu \\ E[Y] &= E[\eta] = \alpha + \beta \mu \\ Var[X] &= Var[\xi] + Var[\delta] = \sigma^2 + \sigma_{\delta}^2 \\ Var[Y] &= Var[\alpha + \beta \xi] + Var[\varepsilon] = \beta^2 \sigma^2 + \sigma_{\varepsilon}^2 \\ Cov[X, Y] &= Cov[\xi, \alpha + \beta \xi] = \beta \sigma^2 \end{split}$$

The method of moments estimating equations can now be found by replacing the

population moments with their sample equivalents

$$\bar{x} = \tilde{\mu} \tag{2.2}$$

$$\bar{y} = \tilde{\alpha} + \tilde{\beta}\tilde{\mu} \tag{2.3}$$

$$s_{xx} = \tilde{\sigma}^2 + \tilde{\sigma}_{\delta}^2 \tag{2.4}$$

$$s_{yy} = \tilde{\beta}^2 \tilde{\sigma}^2 + \tilde{\sigma}_{\varepsilon}^2 \tag{2.5}$$

$$s_{xy} = \hat{\beta}\tilde{\sigma}^2 \tag{2.6}$$

Here, a tilde is placed over the symbol for a parameter to denote the method of moments estimator. From equations (2.4), (2.5) and (2.6) it can be seen that there is a hyperbolic relationship between the method of moments estimators. This was called the Frisch hyperbola by van Montfort [16].

$$(s_{xx} - \tilde{\sigma}_{\delta}^2)(s_{yy} - \tilde{\sigma}_{\varepsilon}^2) = (s_{xy})^2$$

This is a useful equation as it relates pairs of estimates $(\tilde{\sigma}_{\delta}^2, \tilde{\sigma}_{\varepsilon}^2)$ to the data in question.

One of the main problems with fitting an errors in variables model is that of identifiability. It can be seen from equations (2.2), (2.3), (2.4), (2.5) and (2.6) a unique solution cannot be found for the parameters; there are five equations, but six unknown parameters. A way to proceed with this method is to assume that there is some prior knowledge of the parameters that enables a restriction to be imposed. The method of moments equations under this restriction can then be readily solved. Other than this, additional estimating equations may be derived by deriving equations based on the higher moments.

There is a comparison with this identifiability problem and the maximum likelihood approach. The only tractable assumption to obtain a maximum likelihood solution is to assume that the distributions of ξ , δ and ε are all Normal. Otherwise, the algebraic manipulation required becomes an enormous task. Further details will be presented in Gillard [8]. If all the distributions are assumed Normal, this leads to the bivariate random variable (x, y) having a bivariate Normal distribution. This distribution has five parameters, and the maximum likelihood estimators for these parameters are identical to the method of moments estimators based on the moment equations (2.2), (2.3), (2.4), (2.5), and (2.6) above. In this case therefore it is not possible to find unique solutions to the likelihood equations without making an additional assumption, effectively restricting the parameter space. A fuller treatment of restrictions on the parameter space and method of moments estimators can be found in Gillard and Iles [9].

2.2 Estimation of the Linear Structural Model

It is possible to write estimators for the parameters of the linear structural model in terms of β and first or second order moments. Once an estimator for β has been obtained, the following equations can be used to estimate the remaining parameters.

Equation (2.2) immediately yields the intuitive estimator for μ

$$\tilde{\mu} = \bar{x} \tag{2.7}$$

The estimators for the remaining parameters can be expressed as functions of the slope, β , and other sample moments. An estimator for the intercept may be found by substituting (2.2) into (2.3) and rearranging to give

$$\tilde{\alpha} = \bar{y} - \tilde{\beta}\bar{x} \tag{2.8}$$

This shows, just as in simple linear regression, that the errors in variables regression line also passes through the centroid (\bar{x}, \bar{y}) of the data.

Equation (2.6) gives

$$\tilde{\sigma}^2 = \frac{s_{xy}}{\tilde{\beta}} \tag{2.9}$$

with $\tilde{\beta}$ and s_{xy} sharing the same sign.

If the error variance σ_{δ}^2 is unknown, it may be estimated using (2.4)

$$\tilde{\sigma}_{\delta}^2 = s_{xx} - \tilde{\sigma}^2 \tag{2.10}$$

Finally, if the error variance σ_{ε}^2 is unknown, it may be estimated using (2.5)

$$\tilde{\sigma}_{\varepsilon}^2 = s_{yy} - \tilde{\beta}^2 \tilde{\sigma}^2 \tag{2.11}$$

In order to ensure that the estimators for the variances are non negative, admissibility conditions must be placed on the equations. The straightforward conditions are included below

$$s_{xx} > \sigma_{\delta}^2$$

$$s_{yy} > \sigma_{\varepsilon}^2.$$

More precisely, the estimate of the slope must lie between the slopes of the regression line of y on x and that of x on y for variance estimators using (2.4), (2.5) and (2.6) to be non negative.

As stated in Gillard and Iles [9], there is a variety of estimators for the slope. This technical paper will concentrate on the asymptotic theory regarding some of these estimators. The estimators whose variances are derived here will be stated later, but for more detail concerning practical advice and also the derivation of the estimators, the reader is once again referred to Gillard and Iles [9].

Chapter 3 Asymptotics for the Method of

Moments

A common misunderstanding regarding the method of moments is that there is a lack of asymptotic theory associated with the method. This however is not true. Cramer [4] and subsequently other authors such as Bowman and Shenton [1] detailed an approximate method commonly known as the delta method (or the method of statistical differentials) to obtain expressions for variances and covariances of functions of sample moments. The method is sometimes described in statistics texts, for example DeGroot [5], and is often used in linear models to derive a variance stabilisation transformation (see Draper and Smith [6]). The delta method is used to approximate the expectations, and hence also the variances and covariances of functions of random variables by making use of a Taylor series expansion about the expected values. The motivation of the delta method is included below.

Consider a first order Taylor expansion of a function of a sample moment x, f(x) where $E[x] = \mu$.

$$f(x) \simeq f(\mu) + (x - \mu)f'(\mu)$$
 (3.1)

Upon taking the expectation of both sides of (3.1) the usual approximation

$$E[f(x)] \simeq f(\mu)$$

is found. Additionally,

$$Var[f(x)] = E\left[\{f(x) - E[f(x)]\}^2\right] \simeq \{f'(\mu_x)\}^2 E[(x - \mu_x)^2]$$
$$= \{f'(\mu_x)\}^2 Var[x]$$
$$= \left\{\frac{\overline{\partial f}}{\partial x}\right\}^2 Var[x]$$

The notation

$$\left\{ \overline{\frac{\partial f}{\partial x}} \right\} = \frac{\partial f}{\partial x} \bigg|_{x = E[x]}$$

was introduced by Cramer to denote a partial derivative evaluated at the expected values of the sample moments.

This can be naturally extended to functions of more than one sample moment. For a function f(x, y)

$$Var[f(x,y)] \simeq \left\{\overline{\frac{\partial f}{\partial x}}\right\}^2 Var[x] + \left\{\overline{\frac{\partial f}{\partial y}}\right\}^2 Var[y] + 2\left\{\overline{\frac{\partial f}{\partial x}}\right\} \left\{\overline{\frac{\partial f}{\partial y}}\right\} Cov[x,y]$$

and for a function of p sample moments, x_1, \ldots, x_p ,

$$Var[f(x_1,\ldots,x_p)] \simeq \nabla^T V \nabla$$

where

$$\nabla^T = \left(\overline{\frac{\partial f}{\partial x_1}}, \dots, \overline{\frac{\partial f}{\partial x_p}}\right)$$

is the vector of derivatives with each sample moment substituted for its expected value, and

$$V = \begin{pmatrix} Var[x_1] & Cov[x_1, x_2] & \dots & Cov[x_1, x_p] \\ \vdots & \ddots & \vdots \\ Cov[x_1, x_p] & Cov[x_2, x_p] & \dots & Var[x_p] \end{pmatrix}$$

is the $p \times p$ matrix containing the variances of and covariances between sample moments.

Covariances between functions of sample moments can be derived in a similar manner.

Chapter 4

Deriving the Variance Covariance Matrices for Restricted Cases

Essentially, use of the method outlined above requires the prior computation of the variance of each relevant sample moment, and the covariances between each sample moment. For each of the restricted cases discussed by Gillard and Iles [9], the following variances and covariances are used. The variances and covariances needed to compute the asymptotics for the higher moment based estimators will be stated later on in this report.

The variances of the first and second order moments are:

$$Var[\bar{x}] \simeq \frac{\sigma^2 + \sigma_{\delta}^2}{n}$$
 (4.1)

$$Var[\bar{y}] \simeq \frac{\beta^2 \sigma^2 + \sigma_{\varepsilon}^2}{n}$$
 (4.2)

$$Var[s_{xx}] \simeq \frac{(\mu_{\xi 4} - \sigma^4) + (\mu_{\delta 4} - \sigma^4_{\delta}) + 4\sigma^2 \sigma^2_{\delta}}{n}$$
(4.3)

$$Var[s_{xy}] \simeq \frac{\beta^2(\mu_{\xi 4} - \sigma^4) + \sigma^2 \sigma_{\varepsilon}^2 + \beta^2 \sigma^2 \sigma_{\delta}^2 + \sigma_{\delta}^2 \sigma_{\varepsilon}^2}{n}$$
$$Var[s_{yy}] \simeq \frac{\beta^4(\mu_{\xi 4} - \sigma^4) + (\mu_{\varepsilon 4} - \sigma_{\varepsilon}^4) + 4\beta^2 \sigma^2 \sigma_{\varepsilon}^2}{n}$$

The covariances between all first and second order moments are:

$$Cov[\bar{x}, \bar{y}] \simeq \frac{\beta \sigma^2}{n}$$

$$Cov[\bar{x}, s_{xx}] \simeq \frac{\mu_{\xi_3} + \mu_{\delta_3}}{n}$$

$$Cov[\bar{y}, s_{xx}] \simeq \frac{\beta \mu_{\xi_3}}{n}$$

$$Cov[\bar{y}, s_{xy}] \simeq \frac{\beta \mu_{\xi_3}}{n}$$

$$Cov[\bar{y}, s_{xy}] \simeq \frac{\beta^2 \mu_{\xi_3}}{n}$$

$$Cov[\bar{y}, s_{yy}] \simeq \frac{\beta^2 \mu_{\xi_3}}{n}$$

$$Cov[\bar{y}, s_{yy}] \simeq \frac{\beta^3 \mu_{\xi_3} + \mu_{\varepsilon_3}}{n}$$

$$Cov[\bar{y}, s_{yy}] \simeq \frac{\beta(\mu_{\xi_4} - \sigma^4) + 2\beta\sigma^2\sigma_{\delta}^2}{n}$$

$$Cov[s_{xx}, s_{yy}] \simeq \frac{\beta^2(\mu_{\xi_4} - \sigma^4)}{n}$$

$$Cov[s_{xy}, s_{yy}] \simeq \frac{\beta^3(\mu_{\xi_4} - \sigma^4) - 2\beta\sigma^2\sigma_{\varepsilon}^2}{n}$$

$$Cov[s_{xy}, s_{yy}] \simeq \frac{\beta^3(\mu_{\xi_4} - \sigma^4) - 2\beta\sigma^2\sigma_{\varepsilon}^2}{n}$$

Expressions (4.1), (4.2) and (4.4) follow from the definition of the linear structural model. To show how these may be derived, the algebra underlying expressions (4.3) and (4.5) shall be outlined. For brevity the notation $\xi_i^* = \xi_i - \bar{\xi}$ and $\eta_i^* = \eta_i - \bar{\eta}$ is introduced.

4.1 Derivation of $Var[s_{xx}]$

Since ξ_i and δ_i are uncorrelated we can write

$$E[s_{xx}] = E\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right] = E\left[\frac{1}{n}\sum_{i=1}^{n}\{(\xi_i^*) + (\delta_i^*)\}^2\right]$$
$$= \frac{1}{n}E\left[\sum_{i=1}^{n}(\xi_i^*)^2 + 2\sum_{i=1}^{n}(\xi_i^*)(\delta_i^*) + \sum_{i=1}^{n}(\delta_i^*)^2\right]$$
$$\simeq \sigma^2 + \sigma_{\delta}^2.$$

The above result also follows from the method of moment estimating equation stated earlier, $s_{xx} = \sigma^2 + \sigma_{\delta}^2$.

$$E[(s_{xx})^{2}] = \frac{1}{n^{2}}E\left[\left\{\sum_{i=1}^{n} (x_{i} - \bar{x})^{2}\right\}^{2}\right]$$

$$= \frac{1}{n^{2}}E\left[\left\{\sum_{i=1}^{n} \left((\xi_{i}^{*}) + (\delta_{i}^{*})\right)^{2}\right\}^{2}\right]$$

$$= \frac{1}{n^{2}}E\left[\sum_{i=1}^{n} \left((\xi_{i}^{*})^{4} + 4(\xi_{i}^{*})^{3}(\delta_{i}^{*}) + 6(\xi_{i}^{*})^{2}(\delta_{i}^{*})^{2} + 4(\xi_{i}^{*})(\delta_{i}^{*})^{3} + (\delta_{i}^{*})^{4}\right)\right]$$

$$+ \frac{1}{n^{2}}E\left[\sum_{i\neq j} \left((\xi_{i}^{*})^{2}(\xi_{j}^{*})^{2} + 2(\xi_{i}^{*})^{2}(\xi_{j}^{*})(\delta_{j}^{*}) + (\xi_{i}^{*})^{2}(\delta_{j}^{*})^{2} + 2(\xi_{i}^{*})(\xi_{j}^{*})(\delta_{j}^{*})^{2}(\delta_{i}^{*})\right]$$

$$+ 4(\xi_{i}^{*})(\xi_{j}^{*})(\delta_{i}^{*})(\delta_{j}^{*}) + 2(\xi_{i}^{*})(\delta_{i}^{*})^{2}(\delta_{j}^{*}) + (\xi_{j}^{*})^{2}(\delta_{i}^{*})^{2} + 2(\xi_{j}^{*})(\delta_{i}^{*})(\delta_{j}^{*})^{2} + (\delta_{i}^{*})^{2}(\delta_{j}^{*})^{2}\right)\right]$$

$$\simeq \frac{1}{n^{2}}\left(n(\mu_{\xi4} + 6\sigma^{2}\sigma_{\delta}^{2} + \mu_{\delta4}) + n(n-1)(\sigma^{4} + 2\sigma^{2}\sigma_{\delta}^{2} + \sigma_{\delta}^{4})\right)$$

Hence it follows that

$$Var[s_{xx}] = E[(s_{xx})^2] - E^2[s_{xx}]$$
$$\simeq \frac{(\mu_{\xi 4} - \sigma^4) + (\mu_{\delta 4} - \sigma^4_{\delta}) + 4\sigma^2 \sigma^2_{\delta}}{n}$$

4.2 Derivation of $Cov[s_{xx}, s_{xy}]$

$$E[s_{xx}s_{xy}] = \frac{1}{n^2} E\left[\sum_{i=1}^n (x_i - \bar{x})^2 \times \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})\right]$$

Now, $(x_i - \bar{x}) = (\xi_i^*) + (\delta_i^*)$ and $(y_i - \bar{y}) = \beta(\xi_i^*) + (\varepsilon_i^*)$. Substituting these into the

above summation, and multiplying out leads to

$$E[s_{xx}s_{xy}] = \frac{1}{n^2}E\left[\sum_{i=1}^n \left(\beta(\xi_i^*)^4 + 2\beta(\xi_i^*)^3(\delta_i^*) + \beta(\xi_i^*)^2(\delta_i^*)^2 + \beta(\xi_i^*)^3(\delta_i^*) + 2\beta(\xi_i^*)^2(\delta_i^*)^2 + \beta(\xi_i^*)^2(\delta_i^*) + 2\beta(\xi_i^*)^2(\delta_i^*)^2 + \beta(\xi_i^*)^2(\delta_i^*) + 2\beta(\xi_i^*)^2(\delta_i^*) + 2\xi_i^*)(\delta_i^*)^2(\varepsilon_i^*) + \xi_i^*)(\delta_i^*)^2(\varepsilon_i^*) + \xi_i^*(\delta_i^*)^2(\varepsilon_i^*) + \xi_i^*)(\delta_i^*)(\varepsilon_i^*) + \frac{1}{n^2}E\left[\sum_{i\neq j} \left(\beta(\xi_i^*)^2(\xi_j^*)^2 + 2\beta(\xi_i^*)^2(\xi_j^*)(\delta_j^*) + \beta(\xi_i^*)^2(\delta_j^*)^2 + \beta(\xi_i^*)(\xi_j^*)^2(\delta_i^*) + 2\beta(\xi_i^*)(\xi_j^*)(\delta_i^*)(\delta_j^*)^2 + \xi_i^*)(\xi_j^*)^2(\varepsilon_i^*) + 2\xi_i^*)(\xi_j^*)(\delta_j^*)(\varepsilon_i^*) + \xi_i^*)(\delta_j^*)(\delta_j^*)(\varepsilon_i^*) + \xi_i^*)(\delta_j^*)^2(\varepsilon_i^*) + \xi_j^*)^2(\delta_j^*)(\varepsilon_j^*) + 2\xi_j^*)(\delta_i^*)(\delta_j^*)(\varepsilon_i^*) + \xi_i^*)(\delta_j^*)^2(\varepsilon_i^*) + \xi_j^*)^2(\delta_j^*)(\varepsilon_j^*) + 2\xi_j^*)(\delta_j^*)(\varepsilon_i^*) + \xi_i^*)(\delta_j^*)^2(\varepsilon_i^*) + \xi_j^*)^2(\delta_j^*)(\varepsilon_j^*) + 2\xi_j^*)(\delta_j^*)(\delta_j^*)(\varepsilon_i^*) + \xi_j^*)^2(\varepsilon_i^*) + \xi_j^*)^2(\delta_j^*)(\varepsilon_j^*) + 2\xi_j^*)(\delta_j^*)(\delta_j^*)(\varepsilon_i^*) + \xi_j^*)^2(\varepsilon_i^*) + \xi_j^*)^2(\delta_j^*)(\varepsilon_j^*) + 2\xi_j^*)(\delta_j^*)(\varepsilon_i^*) + \xi_j^*)^2(\varepsilon_j^*) + \xi_j^*)^2(\delta_j^*)(\delta_j^*)(\varepsilon_j^*) + \xi_j^*)^2(\delta_j^*)(\delta_j^*)(\delta_j^*) + 2\xi_j^*)(\delta_j^*)(\delta_j^*)(\varepsilon_j^*) + \xi_j^*)^2(\varepsilon_j^*) + \xi_j^*)^2(\delta_j^*) + \xi_j^*)^2(\delta_j^*)(\delta_j^*)(\delta_j^*) + 2\xi_j^*)(\delta_j^*)(\delta_j^*)(\delta_j^*) + \xi_j^*)^2(\delta_j^*) + \xi_j^*)^2(\delta_j^*)(\delta_j^*) + \xi_j^*)^2(\delta_j^*)(\delta_j^*) + \xi_j^*)^2(\delta_j^*) + \xi_j^*)^2(\delta_$$

Hence,

$$Cov[s_{xx}, s_{xy}] = E[s_{xx}s_{xy}] - E[s_{xx}]E[s_{xy}]$$
$$\simeq \frac{\beta(\mu_{\xi 4} - \sigma^4) + 2\beta\sigma^2\sigma_{\delta}^2}{n}.$$

Chapter 5

Constructing the Variance Covariance Matrices

For each restricted case, and for the estimators of the slope based on the higher moments a variance-covariance matrix can be constructed. As there are six parameters in the linear structural model μ , α , β , σ^2 , σ_{δ}^2 and σ_{ε}^2 the maximum size of the variance covariance matrix is 6×6 . If the parameter space is restricted, then the size of the variance covariance matrix will decrease in accordance with the number of assumed parameters.

It is possible to use the delta method in order to construct 'shortcut' formulae or approximations to enable quicker calculation of each element of the variance covariance matrix. Usually, these shortcut formulae depend on the variability of the slope estimator and the covariance of the slope estimator with a first or second order sample moment. In some cases the variances and covariances do not depend on the slope estimator used, and as a result are robust to the choice of slope estimator. These shortcut formulae are stated below, and repeating the style of the previous section, example derivations will be given. For brevity, the notation $|\Sigma| = \sigma_{\delta}^2 \sigma_{\varepsilon}^2 + \beta^2 \sigma^2 \sigma_{\delta}^2 + \sigma^2 \sigma_{\varepsilon}^2$ is introduced. This is the determinant of the variance covariance matrix of the bivariate distribution of x and y.

5.1 Shortcut Formulae for Variances

Firstly, the shortcut formulae for the variances will be considered. $Var[\tilde{\alpha}]$ will be the example derivation provided.

$$\begin{split} &Var[\tilde{\mu}] = \frac{\sigma^2 + \sigma_{\delta}^2}{n} \\ &Var[\tilde{\alpha}] = \mu^2 Var[\tilde{\beta}] + \frac{\beta^2 \sigma_{\delta}^2 + \sigma_{\varepsilon}^2}{n} + 2\mu(\beta Cov[\bar{x}, \tilde{\beta}] - Cov[\bar{y}, \tilde{\beta}]) \\ &Var[\tilde{\sigma}^2] = \frac{\sigma^4}{\beta^2} Var[\tilde{\beta}] + \frac{|\Sigma| + \beta^2(\mu_{\xi 4} - \sigma^4)}{\beta^2 n} - \frac{2\sigma^2}{\beta^2} Cov[s_{xy}, \tilde{\beta}] \\ &Var[\tilde{\sigma}_{\delta}^2] = \frac{\sigma^4}{\beta^2} Var[\tilde{\beta}] + \frac{|\Sigma| + \beta^2(\mu_{\delta 4} - \sigma_{\delta}^4)}{\beta^2 n} + \frac{2\sigma^2}{\beta} \left(Cov[s_{xx}, \tilde{\beta}] - \frac{Cov[s_{xy}, \tilde{\beta}]}{\beta} \right) \\ &Var[\tilde{\sigma}_{\varepsilon}^2] = \beta^2 \sigma^4 Var[\tilde{\beta}] + 2\beta \sigma^2 (\beta Cov[s_{xy}, \tilde{\beta}] - Cov[s_{yy}, \tilde{\beta}]) + \frac{\beta^2 |\Sigma| + (\mu_{\varepsilon 4} - \sigma_{\varepsilon}^4)}{n} \end{split}$$

Derivation of $Var[\tilde{\alpha}]$

$$\begin{split} Var[\tilde{\alpha}] &= Var[\bar{y} - \beta \bar{x}] = \left\{ \frac{\overline{\partial \tilde{\alpha}}}{\partial \bar{y}} \right\}^2 Var[\bar{y}] + \left\{ \frac{\overline{\partial \tilde{\alpha}}}{\partial \tilde{\beta}} \right\}^2 Var[\tilde{\beta}] + \left\{ \frac{\overline{\partial \tilde{\alpha}}}{\partial \bar{x}} \right\}^2 Var[\bar{x}] \\ &+ 2\left\{ \frac{\overline{\partial \tilde{\alpha}}}{\partial \bar{y}} \right\} \left\{ \frac{\overline{\partial \tilde{\alpha}}}{\partial \tilde{\beta}} \right\} Cov[\bar{y}, \tilde{\beta}] + 2\left\{ \frac{\overline{\partial \tilde{\alpha}}}{\partial \bar{x}} \right\} \left\{ \frac{\overline{\partial \tilde{\alpha}}}{\partial \tilde{\beta}} \right\} Cov[\bar{x}, \tilde{\beta}] \\ &+ 2\left\{ \frac{\overline{\partial \tilde{\alpha}}}{\partial \bar{x}} \right\} \left\{ \frac{\overline{\partial \tilde{\alpha}}}{\partial \bar{y}} \right\} Cov[\bar{x}, \bar{y}] \\ &= \mu^2 Var[\tilde{\beta}] + \frac{\beta^2 \sigma_{\delta}^2 + \sigma_{\varepsilon}^2}{n} + 2\mu(\beta Cov[\bar{x}, \tilde{\beta}] - Cov[\bar{y}, \tilde{\beta}]) \end{split}$$

A similar shortcut formula was provided in the paper by Hood et al [11]. As outlined in Gillard [8], they assumed that $(\xi, \delta, \varepsilon)$ follow a trivariate Normal distribution. They then used the theory of maximum likelihood to obtain the information matrices required for the asymptotic variance covariance matrices for the parameters of the model. Applying various algebraic manipulations to $Var[\tilde{\alpha}]$ they showed that

$$Var[\tilde{\alpha}] = \mu^2 Var[\tilde{\beta}] + \frac{\beta^2 \sigma_{\delta}^2 + \sigma_{\varepsilon}^2}{n}$$

The shortcut formula derived above is a generalisation of that derived by Hood et al. [11] to cope with non-Normal ξ . Indeed, if $(\xi, \delta, \varepsilon)$ do follow a trivariate Normal distribution, then as $\tilde{\beta}$ is a function only of second order moments (or higher), $\tilde{\beta}$ is statistically independent of the first order sample moments. As a result $Cov[\bar{x}, \tilde{\beta}] = Cov[\bar{y}, \tilde{\beta}] = 0$ and the shortcut formula derived above collapses to that suggested by Hood et al. [11]

5.2 Shortcut Formulae for Covariances

Now, the shortcut formulae for the covariances of $\tilde{\mu}$ with the remaining parameters will be provided.

$$\begin{aligned} Cov[\tilde{\mu}, \tilde{\alpha}] &= -\frac{\beta \sigma_{\delta}^{2}}{n} - \mu Cov[\bar{x}, \tilde{\beta}] \\ Cov[\tilde{\mu}, \tilde{\beta}] &= Cov[\bar{x}, \tilde{\beta}] \\ Cov[\tilde{\mu}, \tilde{\sigma}^{2}] &= \frac{\mu_{\xi 3}}{n} - \frac{\sigma^{2}}{\beta} Cov[\bar{x}, \tilde{\beta}] \\ Cov[\tilde{\mu}, \tilde{\sigma}_{\delta}^{2}] &= \frac{\mu_{\delta 3}}{n} + \frac{\sigma^{2}}{\beta} Cov[\bar{x}, \tilde{\beta}] \\ Cov[\tilde{\mu}, \tilde{\sigma}_{\varepsilon}^{2}] &= -\beta \sigma^{2} Cov[\bar{x}, \tilde{\beta}] \end{aligned}$$

The shortcut formulae for the covariances of $\tilde{\alpha}$ with all other parameters are listed here.

$$\begin{aligned} Cov[\tilde{\alpha}, \tilde{\beta}] &= Cov[\bar{y}, \tilde{\beta}] - \beta Cov[\bar{x}, \tilde{\beta}] - \mu Var[\tilde{\beta}] \\ Cov[\tilde{\alpha}, \tilde{\sigma}^2] &= \frac{\mu \sigma^2}{\beta} Var[\tilde{\beta}] + \sigma^2 \left(Cov[\bar{x}, \tilde{\beta}] - \frac{Cov[\bar{y}, \tilde{\beta}]}{\beta} \right) - \frac{\mu}{\beta} Cov[s_{xy}, \tilde{\beta}] \\ Cov[\tilde{\alpha}, \tilde{\sigma}^2_{\delta}] &= \frac{\mu}{\beta} Cov[s_{xy}, \tilde{\beta}] - \frac{\beta \mu_{\delta 3}}{n} - \mu Cov[s_{xx}, \tilde{\beta}] - \frac{\mu \sigma^2}{\beta} Var[\tilde{\beta}] \\ &- \sigma^2 \left(Cov[\bar{x}, \tilde{\beta}] - \frac{Cov[\bar{y}, \tilde{\beta}]}{\beta} \right) \\ Cov[\tilde{\alpha}, \tilde{\sigma}^2_{\varepsilon}] &= \frac{\mu_{\varepsilon 3}}{n} + \beta \mu \sigma^2 Var[\tilde{\beta}] + \beta \sigma^2 (\beta Cov[\bar{x}, \tilde{\beta}] - Cov[\bar{y}, \tilde{\beta}]) \\ &+ \mu (\beta Cov[s_{xy}, \tilde{\beta}] - Cov[s_{yy}, \tilde{\beta}]) \end{aligned}$$

The shortcut formulae for the covariances of $\tilde{\beta}$ with the remaining parameters are listed here.

$$Cov[\tilde{\beta}, \tilde{\sigma}^{2}] = \frac{1}{\beta} Cov[s_{xy}, \tilde{\beta}] - \frac{\sigma^{2}}{\beta} Var[\tilde{\beta}]$$

$$Cov[\tilde{\beta}, \tilde{\sigma}^{2}_{\delta}] = Cov[s_{xx}, \tilde{\beta}] - \frac{Cov[s_{xy}, \tilde{\beta}]}{\beta} + \frac{\sigma^{2}}{\beta} Var[\tilde{\beta}]$$

$$Cov[\tilde{\beta}, \tilde{\sigma}^{2}_{\varepsilon}] = Cov[s_{yy}, \tilde{\beta}] - \beta Cov[s_{xy}, \tilde{\beta}] - \beta \sigma^{2} Var[\tilde{\beta}]$$

The shortcut formulae for the covariances of $\tilde{\sigma}^2$ with the remaining parameters are listed here.

$$Cov[\tilde{\sigma}^{2}, \tilde{\sigma}_{\delta}^{2}] = -\frac{\sigma^{4}}{\beta^{2}} Var[\tilde{\beta}] + \frac{\sigma^{2}}{\beta} \left(\frac{2}{\beta} Cov[s_{xy}, \tilde{\beta}] - Cov[s_{xx}, \tilde{\beta}] \right) + \frac{|\Sigma| - 2\sigma^{2}\sigma_{\varepsilon}^{2} - 2\sigma_{\delta}^{2}\sigma_{\varepsilon}^{2}}{\beta^{2}n}$$
$$Cov[\tilde{\sigma}^{2}, \tilde{\sigma}_{\varepsilon}^{2}] = \sigma^{4} Var[\tilde{\beta}] - \frac{\sigma^{2}}{\beta} Cov[s_{yy}, \tilde{\beta}] + \frac{|\Sigma| - 2\beta^{2}\sigma^{2}\sigma_{\delta}^{2} - 2\sigma_{\delta}^{2}\sigma_{\varepsilon}^{2}}{n}$$

Finally, the covariance between the error variance estimates is

$$Cov[\tilde{\sigma}_{\delta}^{2}, \tilde{\sigma}_{\varepsilon}^{2}] = -\sigma^{4} Var[\tilde{\beta}] + \frac{\sigma^{2}}{\beta} Cov[s_{yy}, \tilde{\beta}] - \beta\sigma^{2} Cov[s_{xx}, \tilde{\beta}] + \frac{|\Sigma| - 2\beta^{2}\sigma^{2}\sigma_{\delta}^{2} - 2\sigma^{2}\sigma_{\varepsilon}^{2}}{n}.$$

Again, an example derivation is provided.

Derivation of $Cov[\tilde{\beta}, \tilde{\sigma^2}]$ We have that

$$\tilde{\sigma}^2 = \frac{s_{xy}}{\tilde{\beta}}$$

A first order Taylor expansion of $\tilde{\sigma^2}$ around the expected values of s_{xy} and $\tilde{\beta}$ is

$$\tilde{\sigma}^2 = \sigma^2 + (s_{xy} - \beta \sigma^2) \frac{1}{\beta} - (\tilde{\beta} - \beta) \frac{\sigma^2}{\beta}.$$

Hence,

$$Cov[\tilde{\beta}, \tilde{\sigma}^2] = E[(\tilde{\beta} - \beta)(\tilde{\sigma}^2 - \sigma^2)] = \frac{1}{\beta}Cov[s_{xy}, \tilde{\beta}] - \frac{\sigma^2}{\beta}Var[\tilde{\beta}].$$

5.3 Description of Variance Covariance Matrices for Restricted Cases

The complete asymptotic variance covariance matrices for the different slope estimators under varying assumptions are included in the following pages. For ease of use, the matrices are expressed as the sum of three components, A, B and C. The matrix Aalone is needed if the assumptions are made that ξ , δ and ε all have zero third moments and zero measures of kurtosis. These assumptions would be valid if all three of these variables are normally distributed.

The matrix B gives the additional terms that are necessary if ξ has non zero third moment and a non zero measure of kurtosis. It can be seen that in most cases the B matrices are sparce, needing only adjustment for the terms for $Var[\tilde{\sigma}^2]$ and $Cov[\tilde{\mu}, \sigma^2]$. The exceptions are the cases where the reliability ratio is assumed known, and slope estimators involving the higher moments.

The C matrices are additional terms that are needed if the third moments and measures of kurtosis are non zero for the error terms δ and ε . It is likely that these C matrices will prove of less value to practitioners than the A and B matrices. It is quite possible that a practitioner would not wish to assume that the distribution of the variable ξ is Normal, or even that its third and fourth moments behave like those of a Normal distribution. Indeed, the necessity for this assumption to be made in the likelihood approach may well have been one of the obstacles against a more widespread use of errors in variables methodology. The assumption of Normal like distributions for the error terms, however, is more likely to be acceptable. Thus, in many applications, the C matrix may be ignored.

As a check on the method employed the A matrices were checked with those given by Hood [10] and Hood et al. [11], where a different likelihood approach was used in deriving the asymptotic variance covariance matrices. In all cases exact agreement was found, although some simplification of the algebra has been found to be possible. As discussed earlier, the limitation of the likelihood approach is that it is limited to the case where all random variables are assumed to be Normally distributed. The moments approach described by Gillard and Iles [9] does not have this limitation.

Chapter 6 The Variance Covariance Matrices

This section contains the variance covariance matrices for each of the restricted case slope estimators outlined by Gillard and Iles [9]. These are

- Intercept α known
- Error variance σ_{δ}^2 known
- Error variance σ_{ε}^2 known
- Reliability ratio $\kappa = \frac{\sigma^2}{\sigma^2 + \sigma_{\delta}^2}$ known
- Ratio of the error variances $\lambda = \frac{\sigma_{\varepsilon}^2}{\sigma_{\delta}^2}$ known
- Both variances σ_{δ}^2 and σ_{ε}^2 known

For brevity, the notation $U = \sigma^2 + \sigma_{\delta}^2$, $V = \beta^2 \sigma_{\delta}^2 + \sigma_{\varepsilon}^2$, $e_1 = \mu_{\delta 4} - 3\sigma_{\delta}^4$, $e_2 = \mu_{\varepsilon 4} - 3\sigma_{\varepsilon}^4$ and $e_3 = \beta \lambda \mu_{\delta 3} + \mu_{\varepsilon 3}$ shall be used.

6.1 Intercept α known

The method of moments estimator for the slope based on this assumption is

$$\tilde{\beta} = \frac{\bar{y} - \alpha}{\bar{x}}.$$

Since α is assumed to be known, the variance covariance matrix for $\tilde{\mu}, \tilde{\beta}, \tilde{\sigma}^2, \tilde{\sigma}_{\delta}^2$ and $\tilde{\sigma}_{\varepsilon}^2$ is required.

$$A_{1} = \frac{1}{n} \begin{pmatrix} U & -\frac{\beta\sigma_{\delta}^{2}}{\mu} & \frac{\sigma^{2}\sigma_{\delta}^{2}}{\mu} & -\frac{\sigma^{2}\sigma_{\delta}^{2}}{\mu} & \frac{\beta^{2}\sigma^{2}\sigma_{\delta}^{2}}{\mu} \\ \frac{V}{n\mu^{2}} & -\frac{\sigma^{2}}{\beta\mu^{2}}V & \frac{\sigma^{2}}{\beta\mu^{2}}V & -\frac{\beta\sigma^{2}}{\mu^{2}}V \\ \frac{|\Sigma|}{\beta^{2}} + \frac{\sigma^{4}}{\beta^{2}\mu^{2}}V + 2\sigma^{4} & -\frac{|\Sigma|}{\beta^{2}} - \frac{\sigma^{4}}{\beta^{2}\mu^{2}}V - 2\sigma^{2}\sigma_{\delta}^{2} & -|\Sigma| + \frac{\sigma^{4}}{\mu^{2}}V + 2\sigma^{2}\sigma_{\varepsilon}^{2} \\ \frac{|\Sigma|}{\beta^{2}} + \frac{\sigma^{4}}{\beta^{2}\mu^{2}}V + 2\sigma_{\delta}^{4} & |\Sigma| - \frac{\sigma^{4}}{\mu^{2}}V - 2\beta^{2}\sigma^{2}\sigma_{\delta}^{2} \\ \beta^{2}|\Sigma| + \frac{\beta^{2}\sigma^{4}}{\mu^{2}}V + 2\sigma_{\varepsilon}^{4} \end{pmatrix}$$

$$B_1 = \frac{1}{n} \begin{pmatrix} 0 & 0 & \mu_{\xi 3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & & \mu_{\xi 4} - 3\sigma^4 & 0 & \frac{-2\sigma^2\beta^2\mu_{\xi 3}}{\mu} \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

$$C_{1} = \frac{1}{n} \begin{pmatrix} 0 & 0 & 0 & \mu_{\delta 3} & 0 & \lambda \\ 0 & 0 & -\beta\mu_{\delta 3} & \frac{\mu_{\epsilon 3}}{\mu} \\ 0 & \frac{\sigma^{2}}{\mu}\mu_{\delta 3} & -\frac{\sigma^{2}}{\beta\mu}\mu_{\epsilon 3} \\ \mu_{\delta 4} - 3\sigma^{3}_{\delta} - 2\frac{\sigma^{2}}{\mu}\mu_{\delta 3} & \frac{\beta\sigma^{2}}{\mu}(\beta\mu_{\delta 3} + \mu^{2}\mu_{\epsilon 3}) \\ \mu_{\epsilon 4} - 3\sigma^{3}_{\epsilon} - 2\frac{\beta\sigma^{2}}{\mu}\mu_{\epsilon 3} & \mu_{\epsilon 3} \end{pmatrix}$$

6.2 Error variance σ_{δ}^2 known

The method of moments estimator for the slope based on this assumption is

$$\tilde{\beta} = \frac{s_{xy}}{s_{xx} - \sigma_{\delta}^2}.$$

Since σ_{δ}^2 is assumed known, the variance covariance matrix for $\tilde{\mu}, \tilde{\alpha}, \tilde{\beta}, \tilde{\sigma}^2$ and $\tilde{\sigma}_{\varepsilon}^2$ is required.

$$A_{2} = \frac{1}{n} \begin{pmatrix} U & -\beta\sigma_{\delta}^{2} & 0 & 0 & 0 \\ & \frac{\mu^{2}}{\sigma^{4}}(|\Sigma| + 2\beta^{2}\sigma_{\delta}^{4}) + V & -\frac{\mu}{\sigma^{4}}(|\Sigma| + 2\beta^{2}\sigma_{\delta}^{4}) & \frac{2\mu\beta\sigma_{\delta}^{2}}{\sigma^{2}}U & \frac{2\mu\beta\sigma_{\delta}^{2}}{\sigma^{2}}V \\ & & \frac{1}{\sigma^{4}}(|\Sigma| + 2\beta^{2}\sigma_{\delta}^{4}) & -\frac{2\beta\sigma_{\delta}^{2}}{\sigma^{2}}U & -\frac{2\beta\sigma_{\delta}^{2}}{\sigma^{2}}V \\ & & 2U^{2} & 2\beta^{2}\sigma_{\delta}^{4} \\ & & & 2V^{2} \end{pmatrix}$$

$$B_2 = \frac{1}{n} \begin{pmatrix} 0 & 0 & 0 & \mu_{\xi 3} & 0 \\ 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & \mu_{\xi 4} - 3\sigma^4 & 0 \\ & & & 0 \end{pmatrix}$$

$$C_{2} = \frac{1}{n} \begin{pmatrix} 0 & \frac{\beta\mu\mu_{\delta3}}{\sigma^{2}} & -\frac{\beta\mu_{\delta3}}{\sigma^{2}} & \mu_{\delta3} & \beta^{2}\mu_{\delta3} & \\ & -\frac{2\beta^{2}\mu\mu_{\delta3}}{\sigma^{2}} & \frac{\beta^{2}\mu_{\delta3}}{\sigma^{2}} & -\beta\mu_{\delta3} & \mu_{\varepsilon3} - \beta^{3}\mu_{\delta3} & \\ & & \frac{\beta^{2}}{\sigma^{4}}e_{1} & -\frac{\beta}{\sigma^{2}}e_{1} & -\frac{\beta^{3}}{\sigma^{2}}e_{1} & \\ & & & e_{1} & \beta^{2}e_{1} & \\ & & & & e_{2} + \beta^{4}e_{1} & \end{pmatrix}$$

6.3 Error variance σ_{ε}^2 known

The method of moments estimator for the slope based on this assumption is

$$\tilde{\beta} = \frac{s_{yy} - \sigma_{\varepsilon}^2}{s_{xy}}.$$

Since σ_{ε}^2 is assumed known, the variance covariance matrix for $\tilde{\mu}, \tilde{\alpha}, \tilde{\beta}, \tilde{\sigma}^2$ and $\tilde{\sigma}_{\delta}^2$ is required. For brevity, the notation $W = (\beta^2 |\Sigma| + 2\sigma_{\varepsilon}^2) + V$ is introduced.

$$A_{3} = \frac{1}{n} \begin{pmatrix} U & -\beta\sigma_{\delta}^{2} & 0 & 0 & 0 \\ & \frac{\mu^{2}}{\beta^{2}\sigma^{4}}W & -\frac{\mu}{\beta^{2}\sigma^{4}}W & \frac{2\mu}{\beta^{3}\sigma^{2}}(\sigma_{\varepsilon}^{2}V + \beta^{4}\sigma^{2}\sigma_{\delta}^{2}) & -\frac{2\mu\sigma_{\varepsilon}^{2}V}{\beta^{3}\sigma^{2}} \\ & & \frac{1}{\beta^{2}\sigma^{4}}W & -\frac{2}{\beta^{3}\sigma^{2}}(\sigma_{\varepsilon}^{2}V + \beta^{4}\sigma^{2}\sigma_{\delta}^{2}) & \frac{2\sigma_{\varepsilon}^{2}V}{\beta^{3}\sigma^{2}} \\ & & & \frac{2}{\beta^{4}}(\beta^{4}U^{2} + V^{2} - 2\beta^{4}\sigma_{\delta}^{4}) & -\frac{2}{\beta^{4}}(\sigma_{\varepsilon}^{2}V + 2\beta^{2}\sigma_{\delta}^{2}\sigma_{\varepsilon}^{2}) \\ & & & \frac{2V^{2}}{\beta^{4}} \end{pmatrix}$$

$$B_3 = \frac{1}{n} \begin{pmatrix} 0 & 0 & 0 & \mu_{\xi 3} & 0 \\ 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & \mu_{\xi 4} - 3\sigma^4 & 0 \\ & & & 0 \end{pmatrix}$$

$$C_{3} = \frac{1}{n} \begin{pmatrix} 0 & 0 & 0 & 0 & \mu_{\delta 3} \\ -\frac{2\mu\mu_{\varepsilon 3}}{\beta\sigma^{2}} & \frac{\mu_{\varepsilon 3}}{\beta\sigma^{2}} & -\frac{\mu_{\varepsilon 3}}{\beta^{2}} & \frac{\mu_{\varepsilon 3}}{\beta^{2}} - \beta\mu_{\delta 3} \\ & \frac{1}{\beta^{2}\sigma^{4}}e_{2} & -\frac{1}{\beta^{3}\sigma^{2}}e_{2} & \frac{1}{\beta^{3}\sigma^{2}}e_{2} \\ & & \frac{1}{\beta^{4}}e_{2} & -\frac{1}{\beta^{4}}e_{2} \\ & & & \frac{1}{\beta^{4}}(\beta^{4}e_{1} + e_{2}) \end{pmatrix}$$

6.4 Reliability ratio $\kappa = \frac{\sigma^2}{\sigma^2 + \sigma_{\delta}^2}$ known

The method of moments estimator for the slope based on this assumption is

$$\tilde{\beta} = \frac{s_{xy}}{\kappa s_{xx}}.$$

The variance covariance matrix for $\tilde{\mu}, \tilde{\alpha}, \tilde{\beta}, \tilde{\sigma}^2$ and $\tilde{\sigma}_{\delta}^2$ is required. For brevity, the notation $\varpi = 1 - \kappa$ is introduced.

$$A_{4} = \frac{1}{n} \begin{pmatrix} U & -\beta\sigma_{\delta}^{2} & 0 & 0 & 0 \\ & \mu^{2}\frac{|\Sigma|}{\sigma^{4}} + V & -\mu\frac{|\Sigma|}{\sigma^{4}} & 0 & 2\mu\beta\frac{(1-\kappa)}{\sigma^{2}}|\Sigma| \\ & & \frac{|\Sigma|}{\sigma^{4}} & 0 & -2\beta\frac{(1-\kappa)}{\sigma^{2}}|\Sigma| \\ & & 2\sigma^{4} & -2\beta^{2}\kappa\sigma^{2}\sigma_{\delta}^{2} \\ & & & 4\beta^{2}(1-\kappa)|\Sigma| + 2\sigma_{\varepsilon}^{4} \end{pmatrix}$$

$$B_{4} = \frac{1}{n} \begin{pmatrix} 0 & -\mu \frac{\beta \varpi}{\sigma^{2}} \mu_{\xi 3} & \frac{\beta^{2} \varpi}{\sigma^{2}} \mu_{\xi 3} & \mu_{\xi 3} & -\beta^{2} \varpi \mu_{\xi 3} \\ 0 & 0 & 0 & 0 \\ & \frac{\beta^{2} \varpi^{2}}{\sigma^{4}} (\mu_{\xi 4} - 3\sigma^{4}) & \frac{\beta \kappa \varpi}{\sigma^{2}} (\mu_{\xi 4} - 3\sigma^{4}) & -\frac{\beta^{3} \varpi^{2}}{\sigma^{2}} (\mu_{\xi 4} - 3\sigma^{4}) \\ & \kappa^{2} (\mu_{\xi 4} - 3\sigma^{4}) & -\beta^{2} \kappa \varpi (\mu_{\xi 4} - 3\sigma^{4}) \\ & & \beta^{4} \varpi^{2} (\mu_{\xi 4} - 3\sigma^{4}) \end{pmatrix}$$

6.5 Ratio of the error variances $\lambda = \frac{\sigma_{\varepsilon}^2}{\sigma_{\delta}^2}$ known

The method of moments estimator for the slope based on this assumption is

$$\tilde{\beta} = \frac{(s_{yy} - \lambda s_{xx}) + \sqrt{(s_{yy} - \lambda s_{xx})^2 + 4\lambda(s_{xy})^2}}{2s_{xy}}.$$

The variance covariance matrix for $\tilde{\mu}, \tilde{\alpha}, \tilde{\beta}, \tilde{\sigma}^2$ and $\tilde{\sigma}_{\delta}^2$ is required.

$$A_{5} = \frac{1}{n} \begin{pmatrix} U & \beta \sigma_{\delta}^{2} & 0 & 0 & 0 \\ & \mu^{2} \frac{|\Sigma|}{\sigma^{4}} + V & -\mu \frac{|\Sigma|}{\sigma^{4}} & \frac{2\mu\beta}{(\beta^{2}+\lambda)\sigma^{2}} |\Sigma| & 0 \\ & & \frac{|\Sigma|}{\sigma^{4}} & -\frac{2\beta}{(\beta^{2}+\lambda)\sigma^{2}} |\Sigma| & 0 \\ & & 2\sigma^{4} + \frac{4|\Sigma|}{(\beta^{2}+\lambda)} & -\frac{2\sigma_{\delta}^{2}\sigma_{\varepsilon}^{2}}{(\beta^{2}+\lambda)} \\ & & & 2\sigma_{\delta}^{4} \end{pmatrix}$$

$$B_5 = \frac{1}{n} \begin{pmatrix} 0 & 0 & 0 & \mu_{\xi 3} & 0 \\ 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ & & \mu_{\xi 4} - 3\sigma^4 & 0 \\ & & & 0 \end{pmatrix}$$

$$C_{5} = \frac{1}{n} \begin{pmatrix} 0 & \frac{\mu\lambda\beta}{(\beta^{2}+\lambda)\sigma^{2}}\mu_{\delta3} & -\frac{\lambda\beta}{(\beta^{2}+\lambda)\sigma^{2}}\mu_{\delta3} & \frac{\lambda}{(\beta^{2}+\lambda)}\mu_{\delta3} & \frac{\beta^{2}}{(\beta^{2}+\lambda)}\mu_{\delta3} \\ & -2\frac{\mu\beta}{(\beta^{2}+\lambda)\sigma^{2}}e_{3} & \frac{\beta}{(\beta^{2}+\lambda)\sigma^{2}}e_{3} & -\frac{e_{3}}{(\beta^{2}+\lambda)} & \frac{\beta}{(\beta^{2}+\lambda)}e_{3} - \beta\mu_{\delta3} \\ & \frac{\beta^{2}e_{2}+\lambda^{2}\beta^{2}e_{1}}{(\beta^{2}+\lambda)^{2}\sigma^{4}} & -\frac{(\beta e_{2}+\lambda^{2}\beta e_{1})}{(\beta^{2}+\lambda)^{2}\sigma^{2}} & \frac{\beta e_{2}-\lambda\beta^{3}e_{1}}{(\beta^{2}+\lambda)^{2}\sigma^{2}} \\ & & \frac{e_{2}+\lambda^{2}e_{1}}{(\beta^{2}+\lambda)^{2}} & -\frac{(e_{2}+\lambda\beta^{2}e_{1})}{(\beta^{2}+\lambda)^{2}} \\ & & \frac{e_{2}+\beta^{4}e_{1}}{(\beta^{2}+\lambda)^{2}} \end{pmatrix}$$

6.6 Both variances σ_{δ}^2 and σ_{ε}^2 known

The method of moments estimator for the slope based on this assumption is

$$\tilde{\beta} = sgn(s_{xy})\sqrt{\frac{s_{yy} - \sigma_{\varepsilon}^2}{s_{xx} - \sigma_{\delta}^2}}.$$

The variance covariance matrix for $\tilde{\mu}, \tilde{\alpha}, \tilde{\beta}$ and $\tilde{\sigma}^2$ is required.

$$A_{7} = \frac{1}{n} \begin{pmatrix} U & -\beta\sigma_{\delta}^{2} & 0 & 0 \\ 0 & 0 & 0 \\ & & \frac{|\Sigma|}{\sigma^{4}} + \frac{(\beta^{2}\sigma_{\delta}^{2} - \sigma_{\varepsilon}^{2})}{2\beta^{2}\sigma^{4}} & -\frac{\beta\sigma_{\delta}^{2}}{\sigma^{2}}(U + \sigma^{2}) \\ & & \mu_{\xi 4} - 3\sigma^{4} \end{pmatrix}$$

$$B_7 = \frac{1}{n} \begin{pmatrix} 0 & 0 & 0 & \mu_{\xi 3} \\ 0 & 0 & 0 \\ & 0 & 0 \\ & & \mu_{\xi 4} - 3\sigma^4 \end{pmatrix}$$

$$C_{7} = \frac{1}{n} \begin{pmatrix} 0 & \frac{\mu\beta\mu_{\delta3}}{2\sigma^{2}} & -\frac{\beta\mu_{\delta3}}{2\sigma^{2}} & \mu_{\delta3} \\ & -\frac{\beta^{2}\mu}{\sigma^{2}}\mu_{\delta3} - \frac{\mu}{\beta\sigma^{2}}\mu_{\epsilon3} & \frac{\beta^{2}}{2\sigma^{2}}\mu_{\delta3} + \frac{\mu\epsilon3}{2\beta\sigma^{2}} & -\beta\mu_{\delta3} \\ & & \frac{\beta^{2}}{4\sigma^{4}}e_{1} + \frac{e_{3}}{\beta^{2}\sigma^{4}} & -\frac{\beta}{2\sigma^{2}}e_{1} \\ & & & e_{1} \end{pmatrix}$$

Chapter 7

Variances and Covariances for Higher Moment Estimators

The methodology underlying the derivation of the asymptotic variances and covariances for estimators based on higher moments is identical to that outlined previously. However, the algebraic expressions for the variances and covariances of higher moment based estimators are longer and more cumbersome than those for the restricted parameter space. As a result, the full variance covariance matrices for higher moment estimators will not be reported here. However, the expressions needed to work out the full variance covariance matrices for the slope estimator based on third moments will be provided. These expressions can then be substituted into the shortcut formulae to derive the full variance covariance matrices.

7.1 Estimator based on Third Moments

The estimator for the slope β based on the third order moments derived by Gillard and Iles [9] is

$$\tilde{\beta}_8 = \frac{s_{xyy}}{s_{xxy}}.$$

In order to use the shortcut equations outlined earlier, the quantities $Cov[\bar{x}, \tilde{\beta}_8], Cov[\bar{y}, \tilde{\beta}_8], Cov[\bar{y}, \tilde{\beta}_8], Cov[\bar{x}_{xx}, \tilde{\beta}_8], Cov[s_{xy}, \tilde{\beta}_8]$ and $Cov[s_{yy}, \tilde{\beta}_8]$ are needed. Further, to obtain these quantities, the covariances between each of the first and second order moments $(\bar{x}, \bar{y}, s_{xx}, s_{xy}, s_{yy})$ and the third order moments that occur in $\tilde{\beta}_8$ (s_{xxy}, s_{xyy}) must be obtained. Also, the variances of these third order moments must be obtained, as well as the covariance

between them.

Using the method illustrated in deriving $Var[s_{xx}]$ and $Cov[s_{xx}, s_{xy}]$, the required covariances between the second order and third order moments are:

$$\begin{aligned} Var[s_{xxy}] &= \frac{\beta^2(\mu_{\xi 6} - \mu_{\xi 3}^2) + 6\beta^2\mu_{\xi 4}\sigma_{\delta}^2 + \mu_{\xi 4}\sigma_{\epsilon}^2 + 4\beta^2\mu_{\xi 3}\mu_{\delta 3}}{n} \\ &+ \frac{\beta^2\sigma^2\mu_{\delta 4} + \mu_{\delta 4}\sigma_{\epsilon}^2 + 6\sigma^2\sigma_{\delta}^2\sigma_{\epsilon}^2}{n} \\ Var[s_{xyy}] &= \frac{\beta^4(\mu_{\xi 6} - \mu_{\xi 3}^2) + 6\beta^2\mu_{\xi 4}\sigma_{\epsilon}^2 + \beta^4\mu_{\xi 4}\sigma_{\delta}^2 + 4\beta\mu_{\xi 3}\mu_{\epsilon 3}}{n} \\ &+ \frac{\sigma^2\mu_{\epsilon 4} + \sigma_{\delta}^2\mu_{\epsilon 4} + 6\beta^2\sigma^2\sigma_{\delta}^2\sigma_{\epsilon}^2}{n} \\ Cov[\bar{x}, s_{xxy}] &= \frac{\beta(\mu_{\xi 4} + 3\sigma^2\sigma_{\epsilon}^2)}{n} \\ Cov[\bar{x}, s_{xyy}] &= \frac{\beta^2\mu_{\xi 4} + \sigma_{\delta}^2\sigma_{\epsilon}^2 + \beta^2\sigma^2\sigma_{\delta}^2 + \sigma^2\sigma_{\epsilon}^2}{n} \\ Cov[\bar{y}, s_{xxy}] &= \frac{\beta^2(\mu_{\xi 4} + \sigma_{\delta}^2\sigma_{\epsilon}^2 + \beta^2\sigma^2\sigma_{\delta}^2 + \sigma^2\sigma_{\epsilon}^2)}{n} \\ Cov[\bar{y}, s_{xxy}] &= \frac{\beta((\beta^2\mu_{\xi 4} + 3\sigma^2\sigma_{\epsilon}^2))}{n} \\ Cov[\bar{y}, s_{xxy}] &= \frac{\beta((\mu_{\xi 5} - \sigma^2\mu_{\xi 3}) + 5\beta\mu_{\xi 3}\sigma_{\delta}^2 + 4\beta\sigma^2\mu_{\delta 3})}{n} \\ Cov[s_{xx}, s_{xxy}] &= \frac{\beta^2(\mu_{\xi 5} - \sigma^2\mu_{\xi 3}) + 3\beta^2\mu_{\xi 3}\sigma_{\delta}^2 + \sigma^2(\mu_{\xi 3} + \mu_{\delta 3}) + \beta^2\sigma^2\mu_{\delta 3}}{n} \\ Cov[s_{xy}, s_{xxy}] &= \frac{\beta^2(\mu_{\xi 5} - \sigma^2\mu_{\xi 3}) + \sigma^2\mu_{\epsilon 3} + \sigma_{\delta}^2\mu_{\epsilon 3} + \beta^3\mu_{\xi 3}\sigma_{\delta}^2 + 2\beta\mu_{\xi 3}\sigma_{\epsilon}^2}{n} \\ Cov[s_{xx}, s_{xyy}] &= \frac{\beta^2(\mu_{\xi 5} - \sigma^2\mu_{\xi 3}) + 3\beta^2\mu_{\xi 3}\sigma_{\delta}^2 + \beta^2\sigma^2\mu_{\delta 3} + \mu_{\xi 3}\sigma_{\epsilon}^2 + \mu_{\delta 3}\sigma_{\epsilon}^2}{n} \\ Cov[s_{xy}, s_{xyy}] &= \frac{\beta^2(\mu_{\xi 5} - \sigma^2\mu_{\xi 3}) + 3\beta\mu_{\xi 3}\sigma_{\epsilon}^2 + \sigma^2\mu_{\epsilon 3} + \sigma^3\mu_{\xi 3}\sigma_{\epsilon}^2 + \mu_{\delta 3}\sigma_{\epsilon}^2}{n} \\ Cov[s_{xy}, s_{xyy}] &= \frac{\beta^3(\mu_{\xi 5} - \sigma^2\mu_{\xi 3}) + 3\beta\mu_{\xi 3}\sigma_{\epsilon}^2 + \sigma^2\mu_{\epsilon 3} + \sigma^2\mu_{\epsilon 3} + \sigma^3\mu_{\xi 3}\sigma_{\epsilon}^2}{n} \\ Cov[s_{xy}, s_{xyy}] &= \frac{\beta^3(\mu_{\xi 5} - \sigma^2\mu_{\xi 3}) + 3\beta\mu_{\xi 3}\sigma_{\epsilon}^2 + \sigma^2\mu_{\epsilon 3} + \sigma^2\mu_{\epsilon 3} + \beta^3\mu_{\xi 3}\sigma_{\epsilon}^2}{n} \\ Cov[s_{xy}, s_{xyy}] &= \frac{\beta^4(\mu_{\xi 5} - \sigma^2\mu_{\xi 3}) + 5\beta^2\mu_{\xi 3}\sigma_{\epsilon}^2 + 4\beta\sigma^2\mu_{\epsilon 3}}{n} \\ Cov[s_{xy}, s_{xyy}] &= \frac{\beta^4(\mu_{\xi 5} - \sigma^2\mu_{\xi 3}) + 5\beta^2\mu_{\xi 3}\sigma_{\epsilon}^2 + \alpha\beta^2\mu_{\epsilon 3}\sigma_{\epsilon}^2}{n} \\ + \frac{\mu_{\xi 3}\mu_{\epsilon 3} + \beta^3\mu_{\epsilon 3}\eta_{\epsilon 3} + \beta\beta\sigma^2\sigma_{\delta}^2\sigma_{\epsilon}^2 + \mu_{\delta 3}\mu_{\epsilon 3}\sigma_{\epsilon}^2}{n} \\ + \frac{\mu_{\xi 3}\mu_{\epsilon 3} + \beta^3\mu_{\epsilon 3}\eta_{\epsilon 3} + \beta\beta\sigma^2\sigma_{\delta}^2\sigma_{\epsilon}^2 + \mu_{\delta 3}\mu_{\epsilon 3}\sigma_{\epsilon}^2}{n} \\ + \frac{\mu_{\xi 3}\mu_{\epsilon 3} + \beta^3\mu_{\epsilon 3}\eta_{\epsilon 3} + \beta\beta\sigma^2\sigma_{\epsilon}^2}{n} \\ + \frac{\mu_{\xi 3}\mu_{\epsilon 3} + \beta^3\mu_{\epsilon 3}\eta_{\epsilon 3} + \beta\beta\sigma^2\sigma_{\epsilon}^2}{n} \\ + \frac{\mu_{\xi 3}\mu_{\epsilon 3} + \beta^3\mu_{\epsilon 3}\eta_{\epsilon 3} + \beta\beta\sigma^2\sigma_{\epsilon}^2}{n} \\ + \frac{\mu_{\xi 3}\mu_$$

By using the methodology outlined earlier, we can now obtain the variance of our slope estimator $\tilde{\beta}_8$, and the covariances of our slope estimator with the first and second order

moments.

$$\begin{aligned} Var[\tilde{\beta}_{8}] &= \frac{\beta^{2}\mu_{\xi4}\sigma_{\varepsilon}^{2} + \beta^{4}\mu_{\xi4}\sigma_{\delta}^{2} + 2\beta\mu_{\xi3}\mu_{\varepsilon3} + \sigma^{2}\mu_{\varepsilon4} + \sigma_{\delta}^{2}\mu_{\varepsilon4} - 6\beta^{2}\sigma^{2}\sigma_{\delta}^{2}\sigma_{\varepsilon}^{2}}{\beta^{2}\mu_{\xi3}^{2}n} \\ &+ \frac{2\beta^{4}\mu_{\xi3}\mu_{\delta3} + \beta^{4}\sigma^{2}\mu_{\delta4} + \beta^{2}\mu_{\delta4}\sigma_{\varepsilon}^{2} - 2\beta\mu_{\delta3}\mu_{\varepsilon3}}{\beta^{2}\mu_{\xi3}^{2}n} \\ Cov[\bar{x}, \tilde{\beta}_{8}] &= \frac{\sigma_{\delta}^{2}\sigma_{\varepsilon}^{2} - 2\beta^{2}\sigma^{2}\sigma_{\delta}^{2} + \sigma^{2}\sigma_{\varepsilon}^{2}}{\beta\mu_{\xi3}n} \\ Cov[\bar{y}, \tilde{\beta}_{8}] &= \frac{2\sigma^{2}\sigma_{\varepsilon}^{2} - \sigma_{\delta}^{2}\sigma_{\varepsilon}^{2} - \beta^{2}\sigma^{2}\sigma_{\delta}^{2}}{\mu_{\xi3}n} \\ Cov[s_{xx}, \tilde{\beta}_{8}] &= \frac{-3\beta^{2}\mu_{\xi3}\sigma_{\delta}^{2} - 3\beta^{2}\sigma^{2}\mu_{\delta3} + \mu_{\xi3}\sigma_{\varepsilon}^{2} + \mu_{\delta3}\sigma_{\varepsilon}^{2}}{\beta\mu_{\xi3}n} \\ Cov[s_{xy}, \tilde{\beta}_{8}] &= \frac{2\beta\mu_{\xi3}\sigma_{\varepsilon}^{2} + \sigma^{2}\mu_{\varepsilon3} + \sigma_{\delta}^{2}\mu_{\varepsilon3} - 2\beta^{3}\mu_{\xi3}\sigma_{\delta}^{2} - \beta\mu_{\delta3}\sigma_{\varepsilon}^{2} - \beta^{3}\sigma^{2}\mu_{\delta3}}{\beta\mu_{\xi3}n} \\ Cov[s_{yy}, \tilde{\beta}_{8}] &= \frac{3\beta\mu_{\xi3}\sigma_{\varepsilon}^{2} + 3\sigma^{2}\mu_{\varepsilon3} - \sigma_{\delta}^{2}\mu_{\varepsilon3} - \beta^{3}\mu_{\xi3}\sigma_{\delta}^{2}}{\mu_{\xi3}n} \end{aligned}$$

We now have each of the components needed to use the shortcut formulae to obtain the following variance covariance matrix for the parameters μ , α , β , σ^2 , σ^2_{δ} and σ^2_{ε} when the estimator $\tilde{\beta}_8$ is used.

7.2 Estimator based on Fourth Moments

The estimator for the slope β based on fourth order moments derived earlier is

$$\tilde{\beta}_9 = \sqrt{\frac{s_{xyyy} - 3s_{xy}s_{yy}}{s_{xxxy} - 3s_{xx}s_{xy}}}$$

In order to use the shortcut equations, the quantities $Cov[\bar{x}, \tilde{\beta}_9], Cov[\bar{y}, \tilde{\beta}_9], Cov[s_{xx}, \tilde{\beta}_9], Cov[s_{xx}, \tilde{\beta}_9]$ and $Cov[s_{yy}, \tilde{\beta}_9]$ are needed. Further, to obtain these quantities, the covariances between each of the first and second order moments $(\bar{x}, \bar{y}, s_{xx}, s_{xy}, s_{yy})$ and the fourth order moments that occur in $\tilde{\beta}_9$ (s_{xyyy}, s_{xxxy}) must be obtained. Also, the variances of these fourth order moments must be obtained, as well as the covariance between them. The formulae for these quantities are very lengthy and full details will be given in Gillard [?]. However, there is a potential difficulty.

As can be seen from the shortcut formulae, a key component of the variance covariance matrices is $Var[\tilde{\beta}]$. For the estimator of the slope based on fourth moments, $Var[\tilde{\beta}]$ depends on the sixth moment of ξ . High order moments may be difficult to estimate reliably, so the authors believe further work is needed to establish whether this formula is of practical value. Again, full details will be given by Gillard [8].

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