Operator Algebras and Noncommutative Geometry

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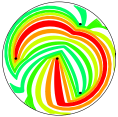
Cardiff University



LMS Graduate Meeting
15 November 2013

Overview

- Gelfand Theorem
- operator algebras
- dimension
- rotation algebra and orbifolds
- Stone-von Neumann theorem

Algebra of continuous functions

$A = C[-1,1] =$ continuous maps $f : [-1,1] \to \mathbb{C}$

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- Spectrum $\quad \mathrm{Spec}(A)$

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recover topology of $[-1,1]$ from $C[-1.1]$

Geland-Naimark Theorem

If A is a Banach $*$-algebra where

$$\|T^*T\| = \|T\|^2, \quad T \in A$$

then A is a C^*-algebra,

— there exists an isometric $*$-homomorphism $\pi : A \to B(\mathcal{H})$ for a Hilbert space $\mathcal{H}$.

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Represent $f \in C[-1,1]$ on $L^2(\mathbb{T})$ by multiplication operators

$$\pi(f) \in B(L^2(\mathbb{T})) \qquad (\pi(f)g)(x) = f(x)g(x)$$

$$A = C[-1,1] \subset B(L^2(\mathbb{T}))$$

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- $H = H^* \in B(\mathcal{H})\,, \quad C^*(H,1) \simeq C(\sigma(H))$

  $\mathrm{Spec}(A) = \sigma(H)$

  $f(H)\,, \quad f \in C(\sigma(H))$

  $x \to exp(itx)\,, \quad U(t) = exp(itH)$

  $U(t)U(s) = U(s+t)$

noncommutative algebra

- points to matrices M_n
  representations are all multiples of $x \to x$
- Spectrum of M_n = singleton $\bullet$
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B = continuous maps from $[-1,1] \to M_2$

$\mathrm{Spec}(B) = [-1,1] \qquad f \to f(x)$

orbifold

$[-1, 1]$ under $\mathbb{Z}_2$ action $x \to -x$

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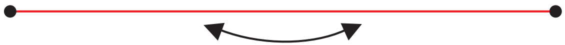

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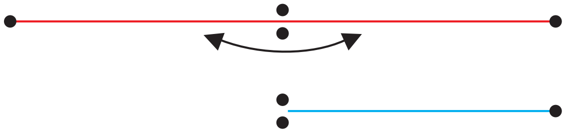

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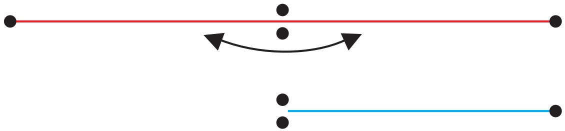

$\mathbb{Z}_2$ action on $B = C([-1, 1], M_2)$

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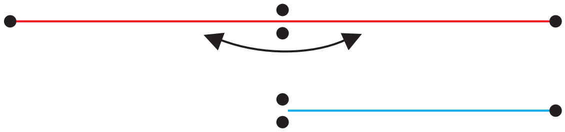

$\mathbb{Z}_2$ action on $B = C([-1, 1], M_2)$

by $x \to -x$ and $U(\quad)U$ where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$f \to g(x) = Uf(-x)U$

orbifold

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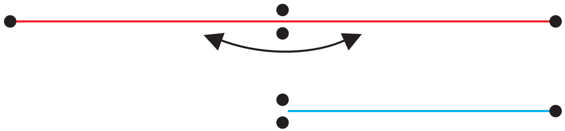

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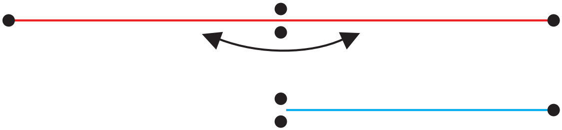

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$\mathrm{Spec}(B^{\mathbb{Z}_2}) =$ 

$$B^{\mathbb{Z}_2} = C^*([-1,1]/\mathbb{Z}_2)$$

the Fermion algebra

$$M_{2^n} \to M_{2^n+2^n} = M_{2^{n+1}} \simeq M_{2^n} \otimes M_2$$

$$a \to \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \simeq a \otimes 1$$

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Iterating

$$\mathbb{C} \to M_2 \to M_4 \to M_8 \to \cdots \to M_{2^n} \to M_{2^{n+1}} \to \cdots$$

$$\bigotimes^n M_2 \simeq M_{2^n} \simeq End(\bigotimes^n \mathbb{C}^2)$$

$$\bigotimes^n M_2 \simeq M_{2^n}$$

Dimensions of Operator Algebras

Projections $e = e^2 = e^*$ in $A \subset B(\mathcal{H})$
correspond to decompositions $\mathcal{H} = e\mathcal{H} \oplus e\mathcal{H}^\perp$

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$$K_0(\otimes_{\mathbb{N}} M_2) = \mathbb{Z}[1/2]$$

discrete rotation matrices

$$V = \begin{pmatrix} 0 & 1 & & & 0 \\ 0 & 0 & & & \\ \cdot & & \cdot & & \\ & & & \cdot & 1 \\ 1 & & & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & & & & 0 \\ 0 & \omega & & & \\ \cdot & & \cdot & & \\ & & & \cdot & \\ 0 & & & 0 & \omega^{n-1} \end{pmatrix}$$

$\omega^n = 1$

$$VUV^* = \omega U$$

$$Ve_r = e_{r-1} \qquad Ue_r = \omega^r e_r$$

rotation algebra

multiplication and translation operators on $L^2(\mathbb{T})$,

$$Uh(z) = zh(z)$$

$$Vh(z) = h(\exp(2\pi i\theta)z)$$

$$VU = \exp(2\pi i\theta)UV$$

- $A_\theta = C^*(U, V)$
  $VUV^{-1} = \exp(2\pi i\theta)U$;

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- if θ is irrational, A_θ is simple, unique
- cf. $C(\mathbb{T}^2)$ when $\theta = 0$.

trace on rotation algebra

$\mathbb{T}^2$ action α on A_θ by

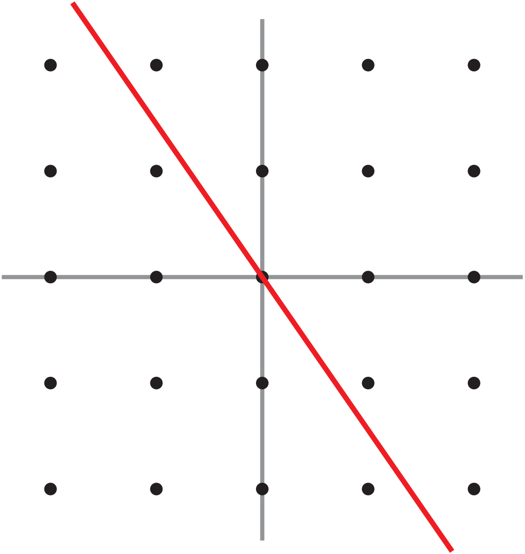
$$(t_1, t_2) \in \mathbb{T}^2$$

$$U \to t_1 U, V \to t_2 V$$

unique trace $\tau = \int \int \alpha_{s,t}\, ds\, dt$,

$$\tau(f(U)V^n) = \int f(t)\, dt\, \delta_{n,0}$$

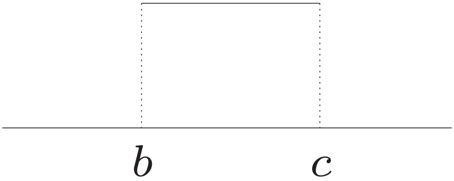
K-group of dimensions generated by trace of projections



$K_0(A_\theta) \cong \mathbb{Z} + \theta\mathbb{Z}$

(m, n) $n + \theta m > 0$

$K_1(A_\theta) \cong \mathbb{Z}^2$, If $\theta = 0$, $K_1(C(\mathbb{T}^2))$ is generated by the coordinate unitary maps $(z_1, z_2) \to z_i$, $i = 1, 2$, i.e. U and V.


b
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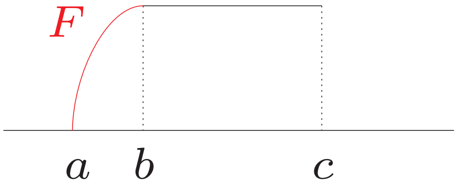
projections in rotation algebras



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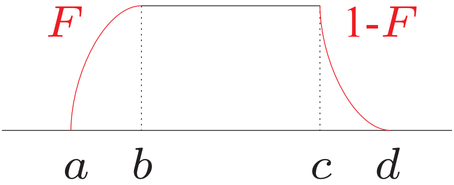

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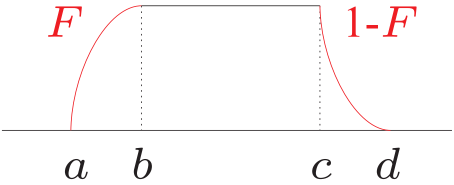

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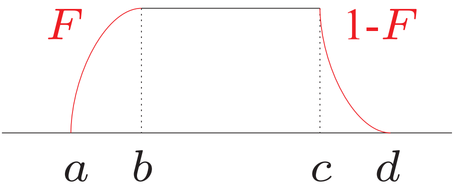
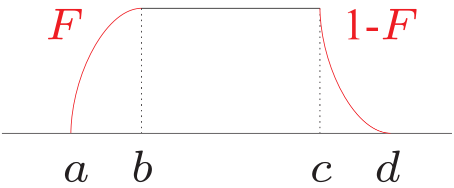


$$e = \chi_{[b,c]} + \begin{pmatrix} F & \sqrt{F(1-F)} \\ \sqrt{F(1-F)} & 1-F \end{pmatrix}$$

projections in rotation algebras



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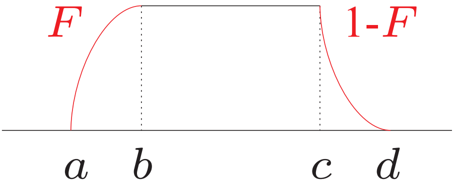
projections in rotation algebras



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$$= V^{-1}g + f + gV \in C^*(U,V) \equiv A_\theta \,,$$

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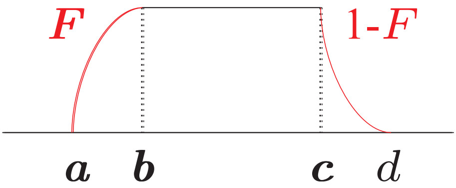
g **continuous** function $= \sqrt{F(1-F)}\chi_{[a,b]}$

projections in rotation algebras



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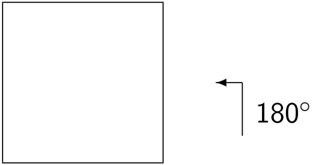
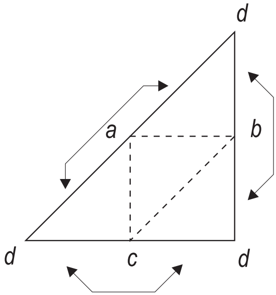
$$\tau(e_\theta) = \int f = \theta$$

The flip $(z_1, z_2) \to (\overline{z}_1, \overline{z}_2)$ on $\mathbb{T}^2$



The orbifold $\mathbb{T}^2/(\mathbb{Z}/2)$ yields a sphere.

toroidal orbifold



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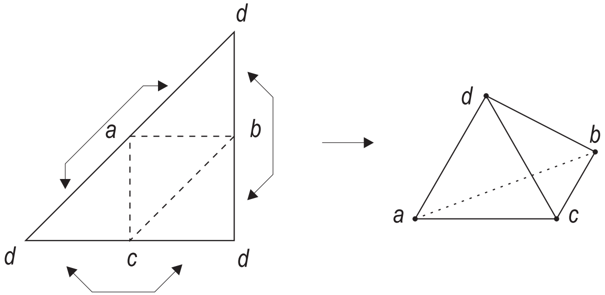

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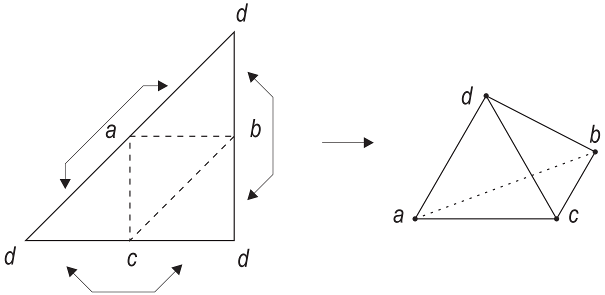

Toroidal orbifold, $\mathbb{T}^2/(\mathbb{Z}/2)$

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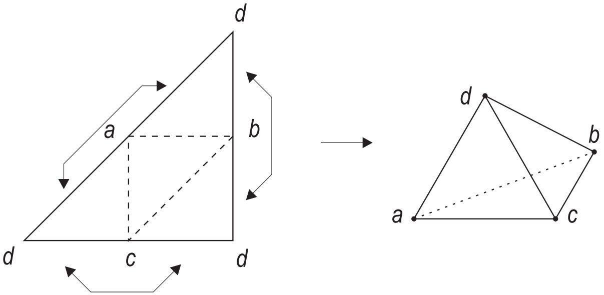

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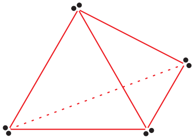

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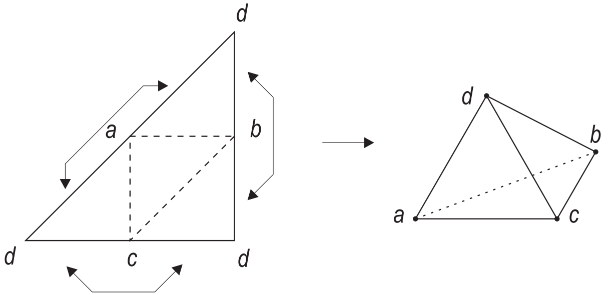

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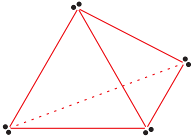


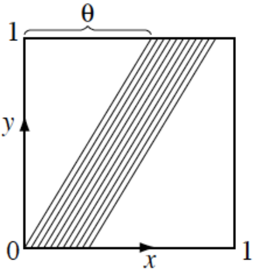
$C(\mathbb{T}^2) \rtimes (\mathbb{Z}/2) = M_2$ valued functions on the sphere, restricted to $\mathbb{C}^2$ at 4 points

non-commutative toroidal orbifold

symmetry $U, V \to U^{-1}, V^{-1}$ preserves relation $VU = e^{2\pi i\theta} UV$

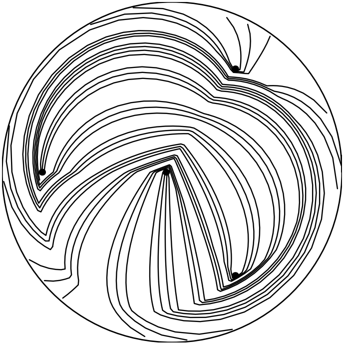
.

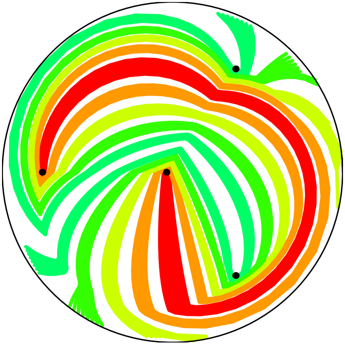
$$A_\theta^{\mathbb{Z}/2} \subset A_\theta \subset A_\theta \rtimes (\mathbb{Z}/2)$$



$$C(\mathbb{T}/\mathbb{Z}) = \mathbb{C} = C(\mathbb{T}^2/\mathbb{R})$$

$$C(\mathbb{T}) \rtimes \mathbb{Z} \sim_{Morita} C(\mathbb{T}^2) \rtimes \mathbb{R}$$





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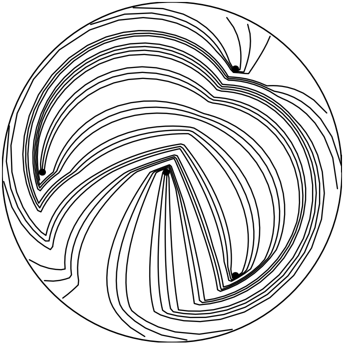
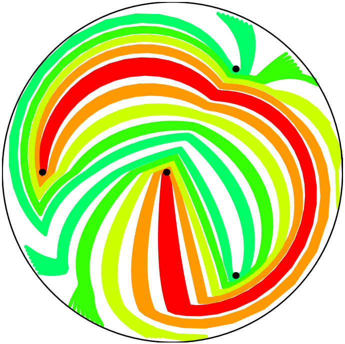
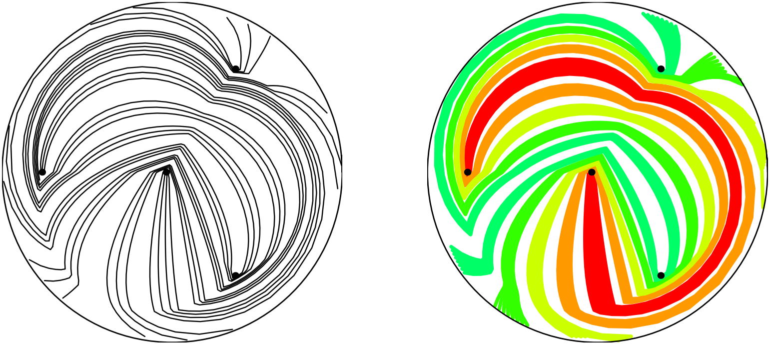




$$A_\theta = lim_{\rightarrow} M_r(C(\mathbb{T})) \oplus M_s(C(\mathbb{T}))$$

non-commutative toroidal orbifold



$$A_\theta = lim_{\rightarrow} M_r(C(\mathbb{T})) \oplus M_s(C(\mathbb{T}))$$

$$A_\theta^{\mathbb{Z}_2} = lim_{\rightarrow} M_{r_1}(\mathbb{C}) \oplus M_{r_2}(\mathbb{C}) \oplus \cdots$$

almost Mathieu operator

$A_\theta^{\mathbb{Z}/2} = C^*(U + U^{-1}, V + V^{-1})$ is AF

Hamiltonians

$$H = U + U^{-1} + \lambda(V + V^{-1})$$

$Uh(z) = zh(z)$, $Vh(z) = h(\exp(2\pi i\theta)z)$.

almost Mathieu operator:

$$f \to f(n+1) + f(n-1) + 2\lambda\cos(2\pi n\theta)f(n)$$

has Cantor spectrum for θ irrational, $\lambda \neq 0$.

Stone-von Neumann Theorem

- $U(m) = U^m, V(n) = V^n$
  $U(m)V(n) = exp(2\pi imn\theta)V(n)U(m)\,,\ m, n \in \mathbb{Z}$
- Continuous version – CCR Canonical Commutation Relations
  $U(t)V(s) = exp(2\pi ist\theta)V(s)U(t)\,,\ s, t \in \mathbb{R}$
  $U(t) = exp(itP), V(s) = exp(isQ)$

Stone-von Neumann Theorem

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  $U(m)V(n) = exp(2\pi imn\theta)V(n)U(m)\,,\ m, n \in \mathbb{Z}$
- Continuous version – CCR Canonical Commutation Relations
  $U(t)V(s) = exp(2\pi ist\theta)V(s)U(t)\,,\ s, t \in \mathbb{R}$
  $U(t) = exp(itP), V(s) = exp(isQ)$
- $PQ - QP = -i$
- irreducible Schrödinger representation is

$$Q = x\,, P = -id/dx \quad on \quad L^2(\mathbb{R})$$

Stone-von Neumann theorem

There is an unique irreducible representation
of the Canonical Commutation Relations.

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Mackey-Rieffel picture:

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$$C_0(G/H) \rtimes G \sim_{Morita} C^*(H)$$

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